Combinatoire expérimentale

Université d'été
Saint-Flour
Août 2007

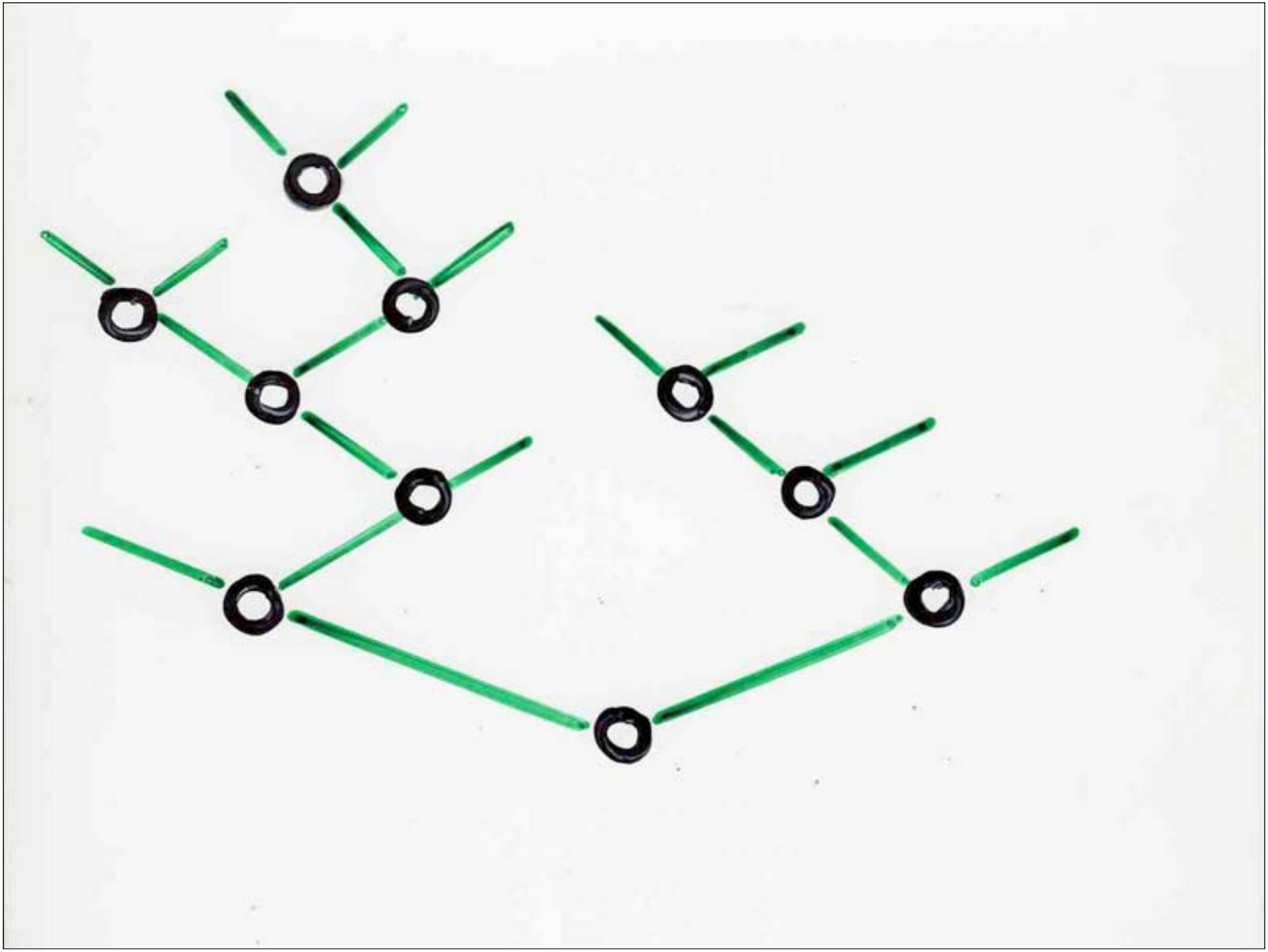
Xavier Viennot
CNRS, LaBRI,
Université Bordeaux 1

Direction générale de l'Enseignement scolaire
© Ministère de l'Éducation nationale

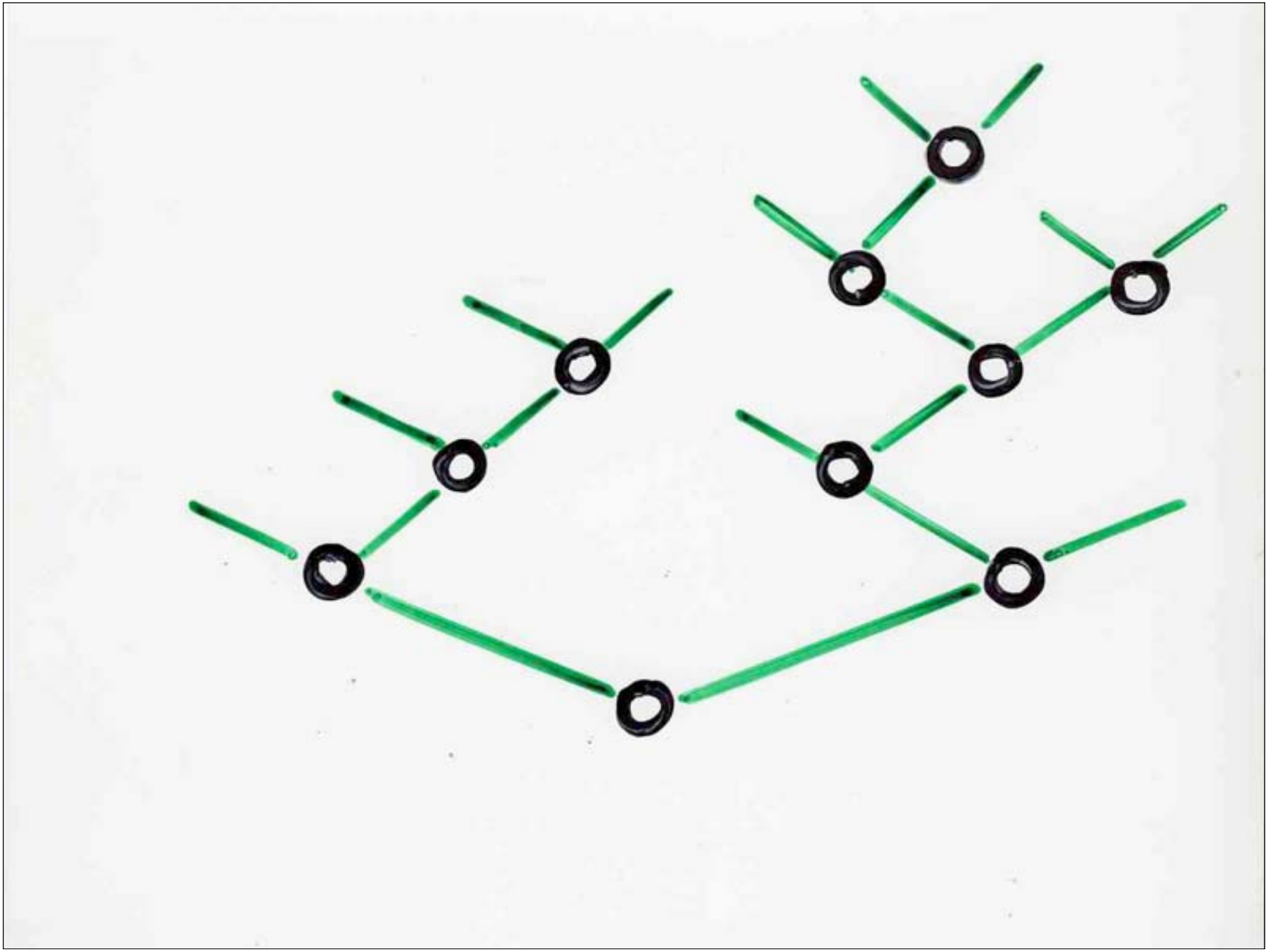
Combinatoire énumérative

$a_n =$?

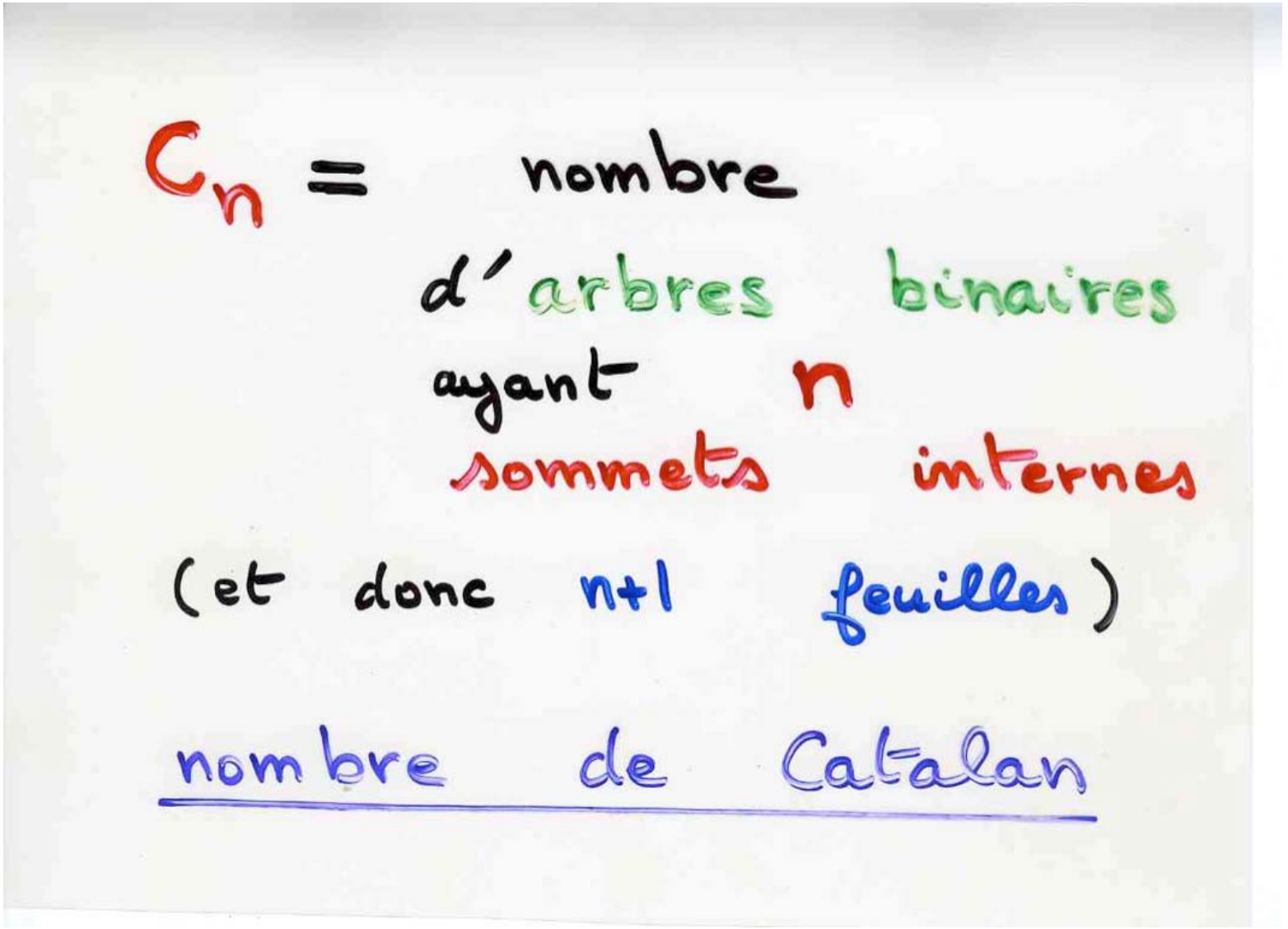
© Ministère de l'Éducation nationale



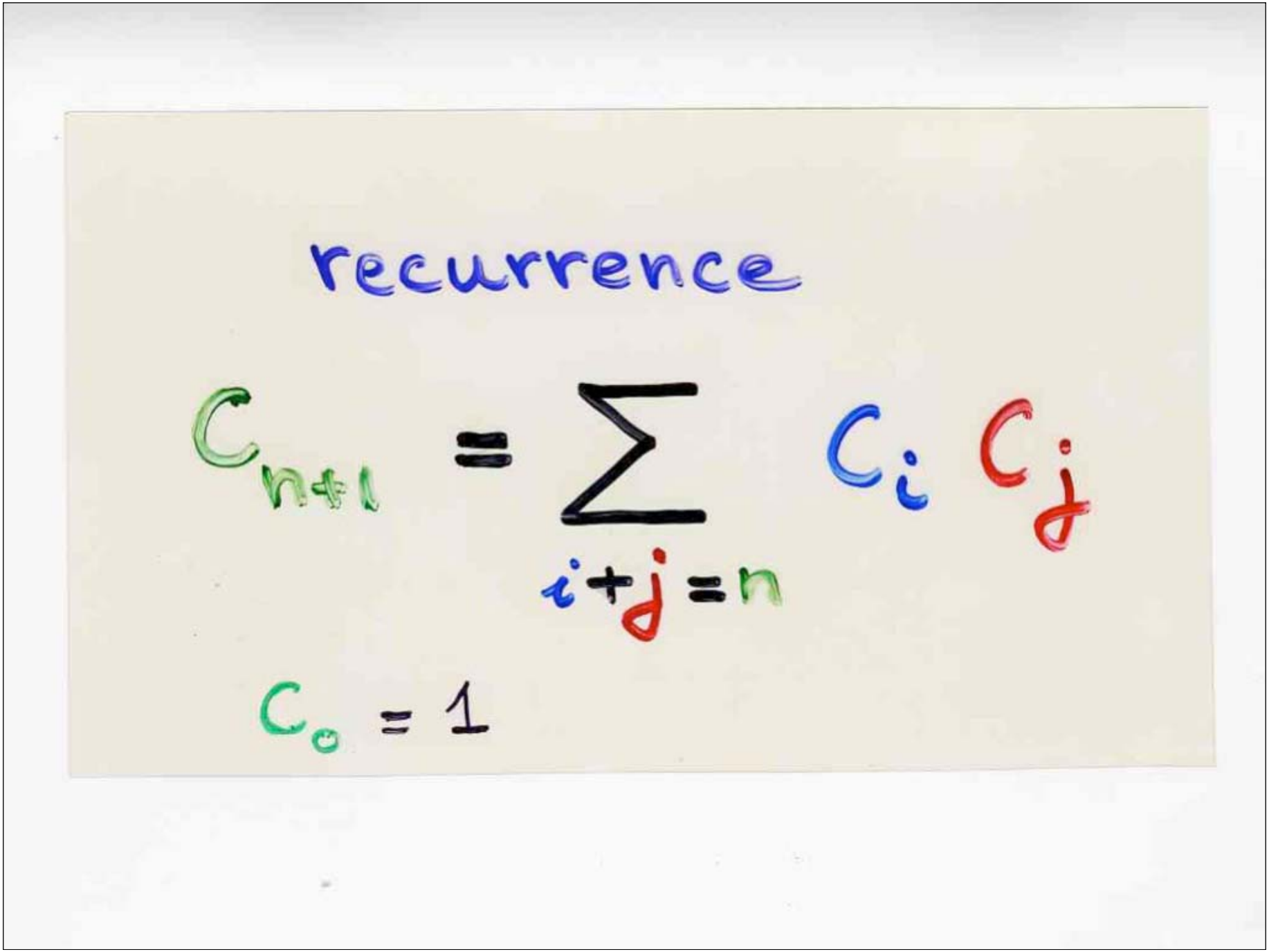
© Ministère de l'Éducation nationale



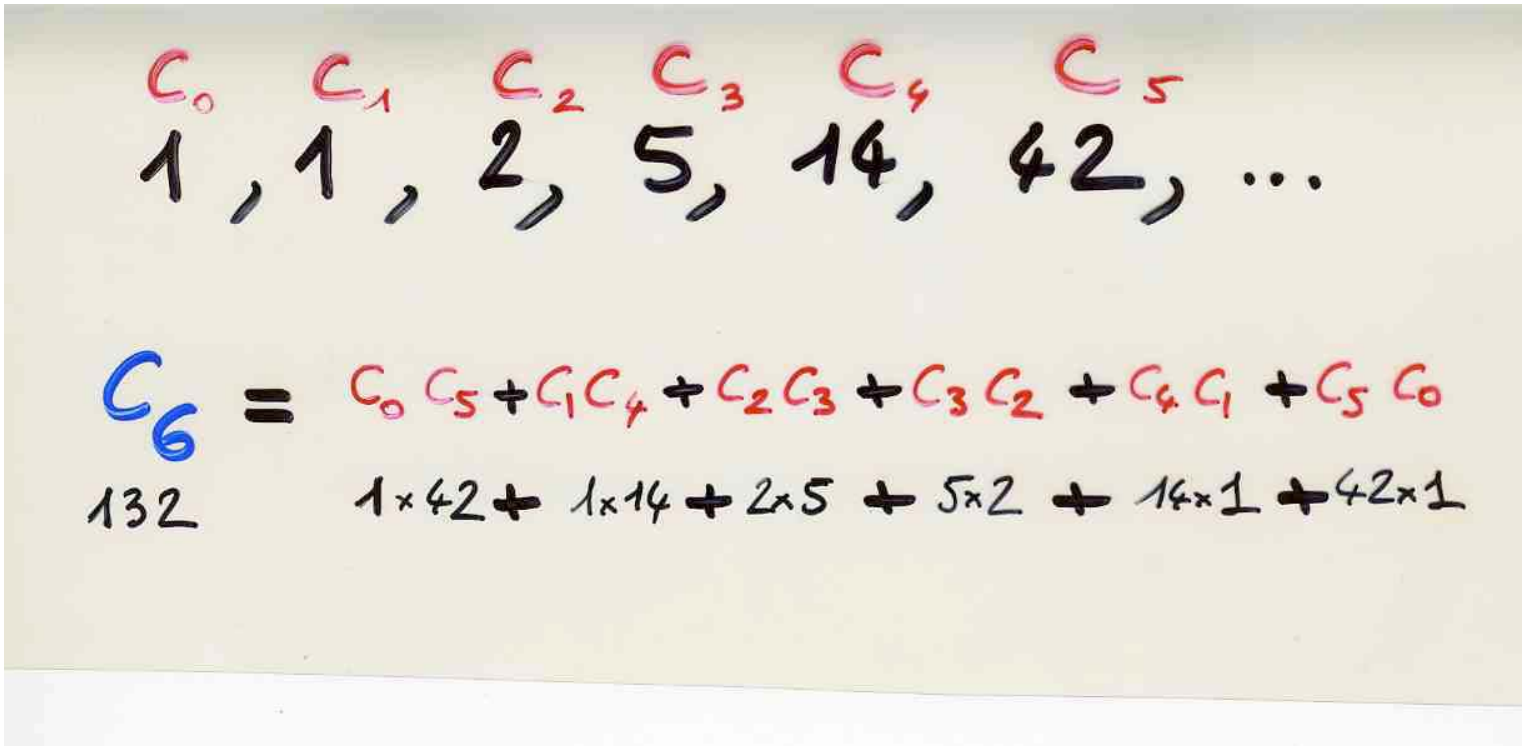
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

© Ministère de l'Éducation nationale

$$C_4 = \frac{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8}{1\times 2\times 3\times 4 \quad 1\times 2\times 3\times 4\times 5}$$

$$= 14$$

© Ministère de l'Éducation nationale

(1838)

Note sur une Équation aux différences finies ;

PAR E. CATALAN.

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \ldots + P_4P_{n-2} + P_3P_{n-1} + P_n, \quad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n} P_n. \quad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3} \cdot \frac{10}{4} \cdot \frac{14}{5} \ldots \frac{4n-6}{n}$$

et comme, dans la question de géométrie qui
équations, on a $P_3 = 1$, nous prendrons simplem

$$P_{n+1} = \frac{2.6.10.14\ldots(4n-6)}{2.3.4.5\ldots n}$$

Le numérateur

$$2.6.10.14\ldots(4n-6) = 2^{n-1}.1.3.5.7.$$
$$= \frac{2^{n-1}.1.2.3.4.5\ldots(2n-2)}{2.4.6.8\ldots(2n-2)} =$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\ldots(2n-2}{2.3.4\ldots n}$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de m lettres, prises p à p ; et si l'on change n en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1} C_{2n,n}, \quad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \quad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{aligned} \frac{1}{n+1} C_{2n,n} = \frac{1}{n} C_{2n-2,n-1} + \frac{1}{n-1} C_{2n-4,n-2} \times \frac{1}{2} C_{3,1} \\ + \frac{1}{n-2} C_{2n-6,n-3} \times \frac{1}{3} C_{4,2} + \ldots + \frac{1}{n} C_{2n-2,n-1}. \end{aligned}\right\} \quad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale ; savoir :

1, 2, 6, 20, 70, 252, 924... ;

ise respectivement par

1, 2, 3, 4, 5, 6, 7... ;

une nouvelle suite de nombres,

1, 1, 2, 5, 14, 42, 132..., (A)

ront de cette propriété :

quelconque de la suite (A) *est égal à la somme des l'on obtient en écrivant au-dessous d'elle-même, et re inverse, la série des termes précédents, et en multi- rmes correspondants des deux séries.*

ple,

$= 1.42 + 1.14 + 2.5 + 5.2 + 14.1 + 42.1.$

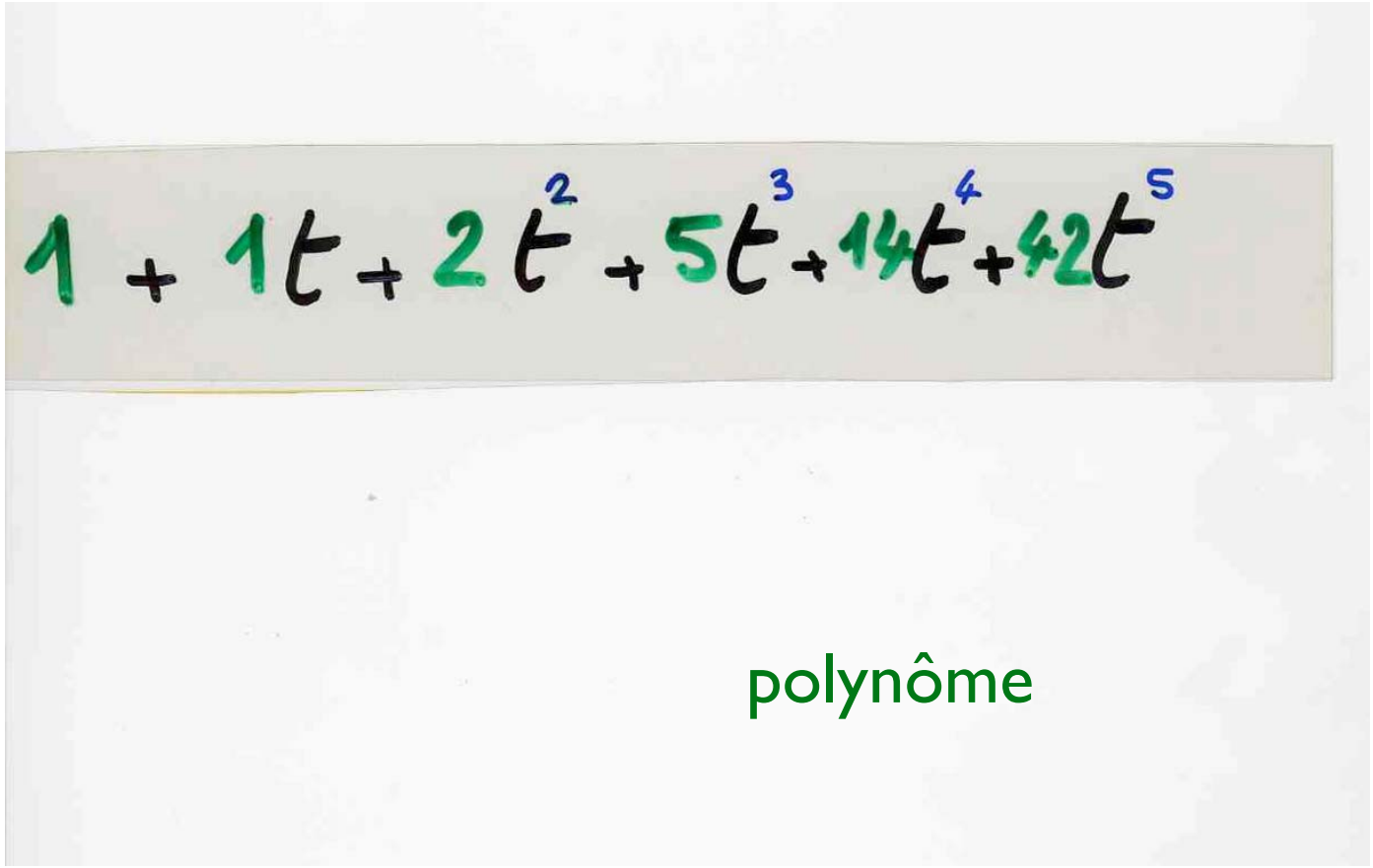
Direction générale de l'Enseignement scolaire
© Ministère de l'Éducation nationale

séries formelles

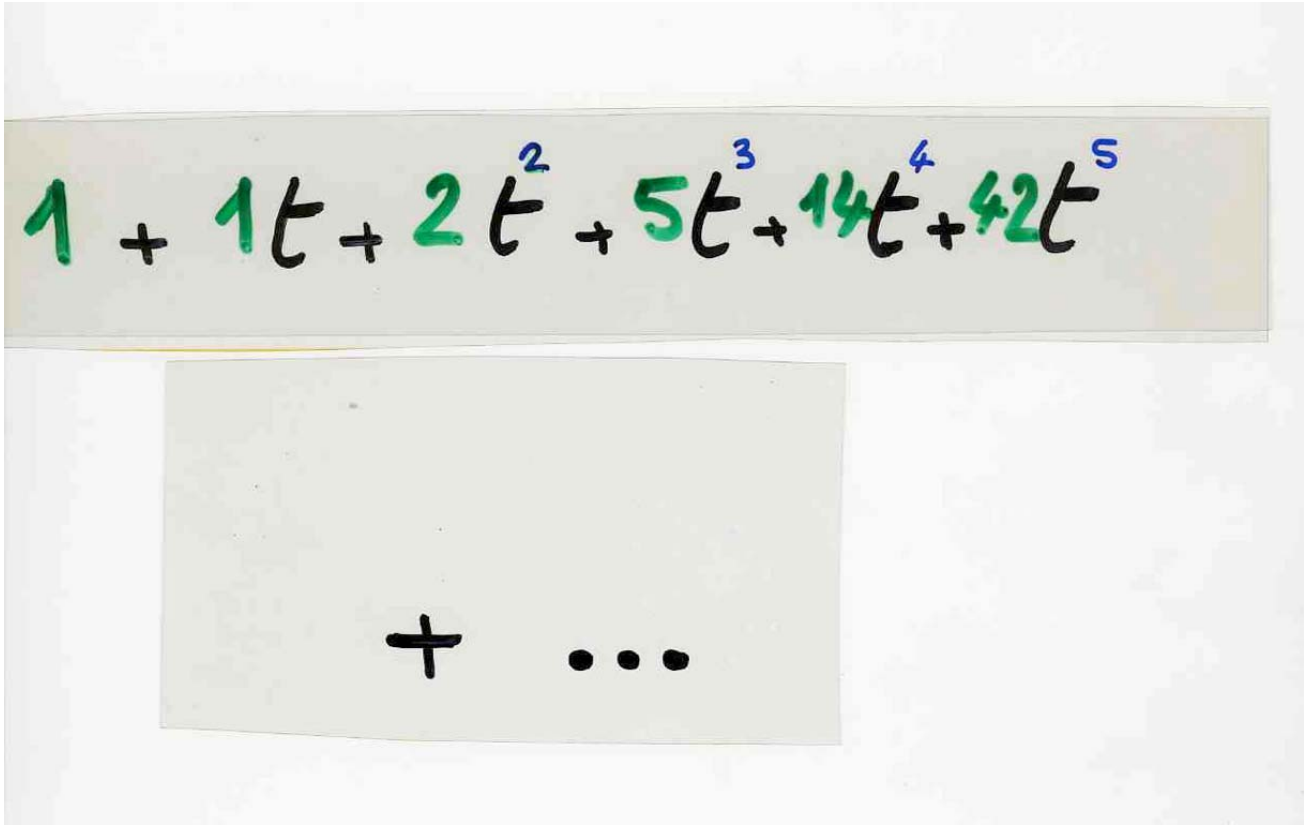
séries génératrices

© Ministère de l'Éducation nationale

© Ministère de l'Éducation nationale

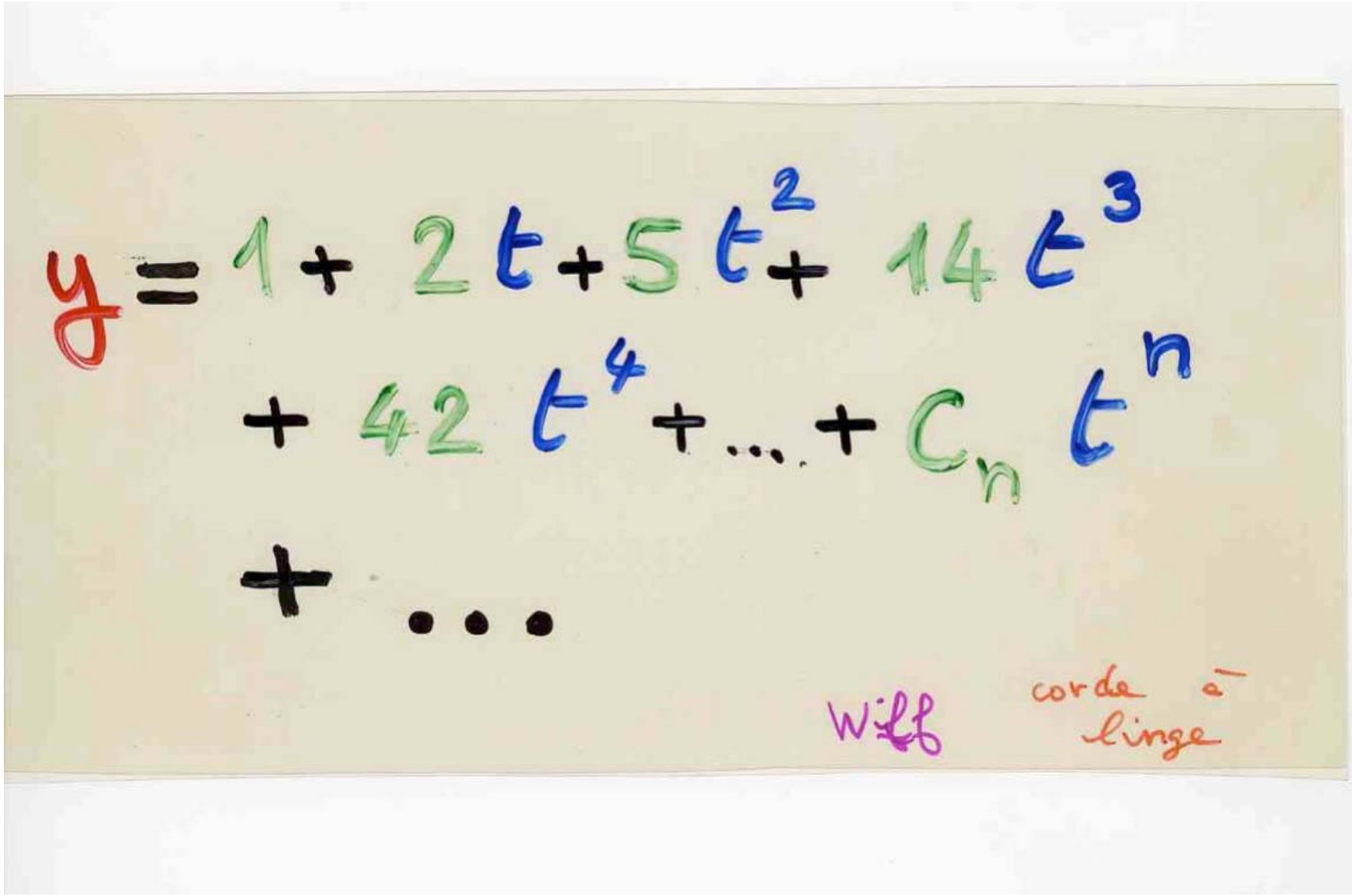


© Ministère de l'Éducation nationale



série formelle

© Ministère de l'Éducation nationale



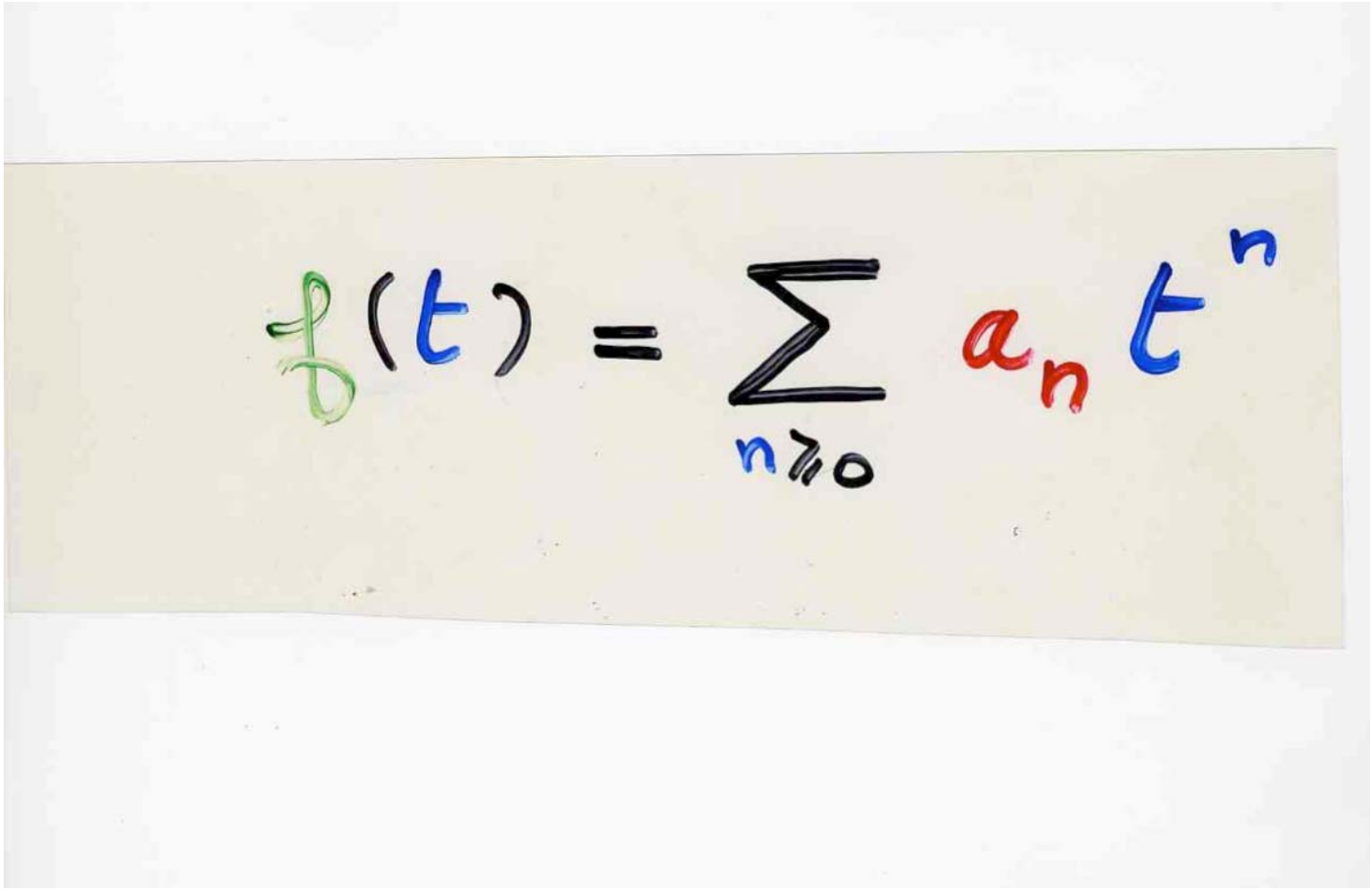
© Ministère de l'Éducation nationale

série génératrice

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots$$

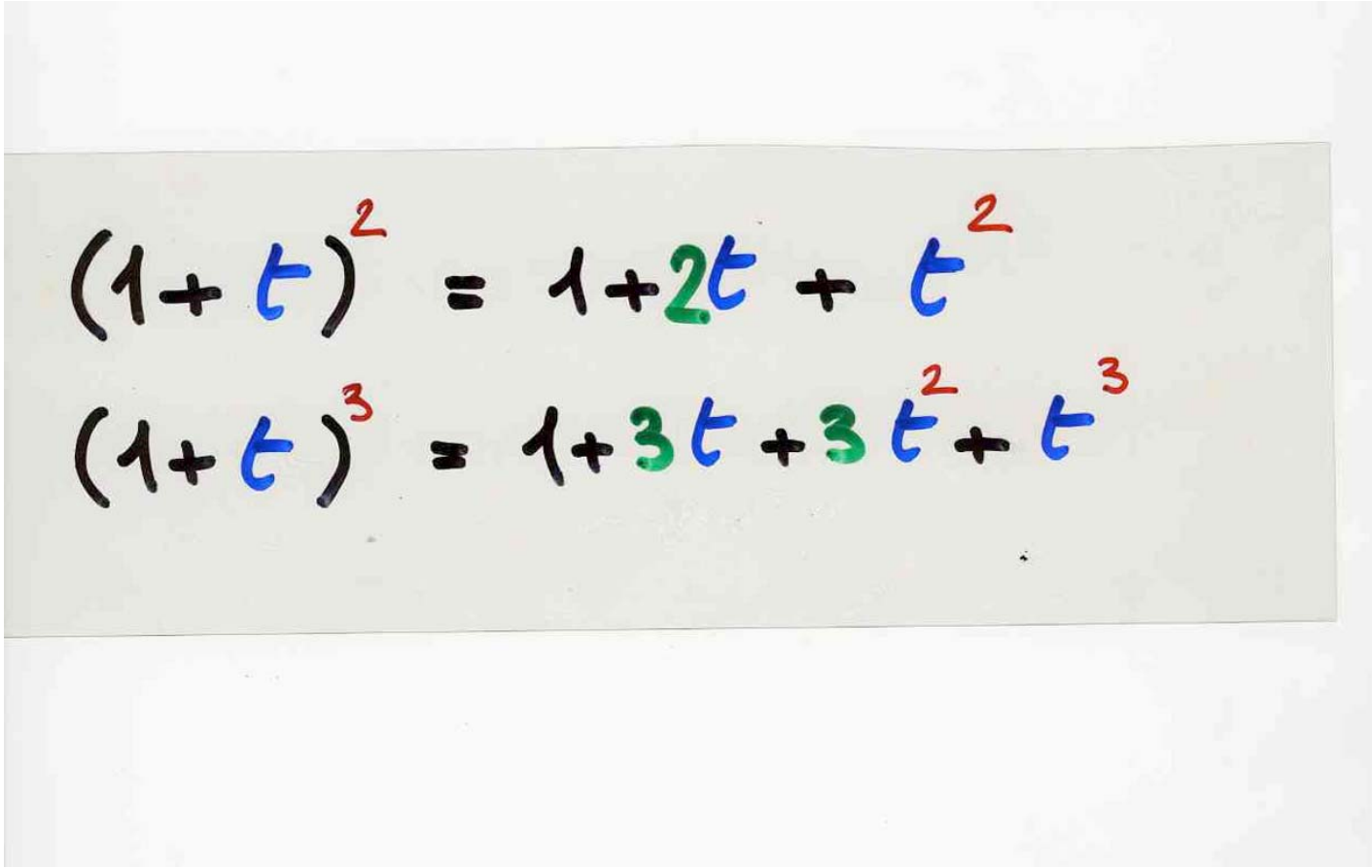
$$\ldots + a_n t^n + \ldots\ldots$$

© Ministère de l'Éducation nationale



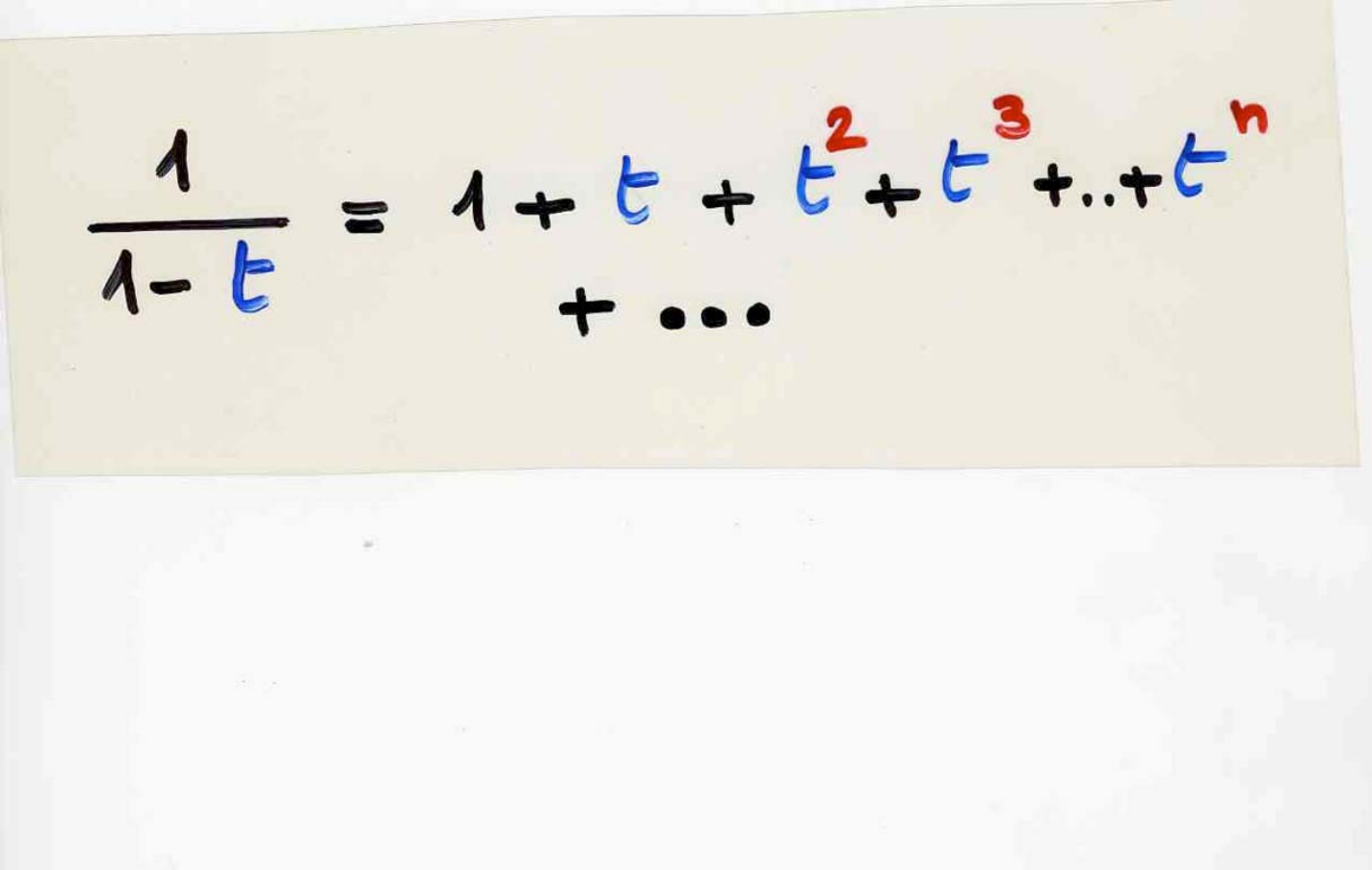
© Ministère de l'Éducation nationale

algèbre des polynômes



© Ministère de l'Éducation nationale

algèbre des séries formelles



© Ministère de l'Éducation nationale

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots + t^n + \ldots$$

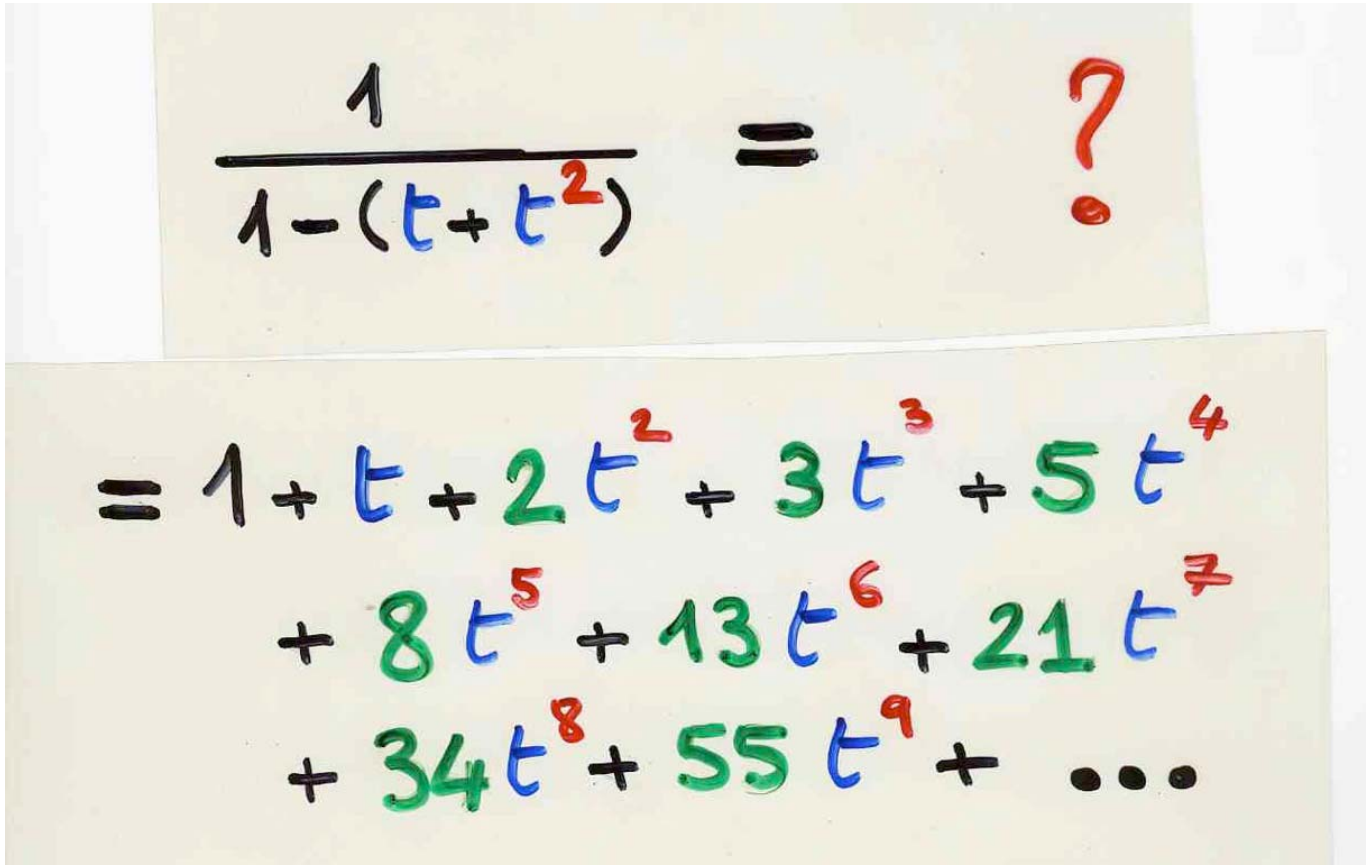
$$(1 + t + t^2 + \ldots + t^n + \ldots) \times (1 - t)$$

© Ministère de l'Éducation nationale

exercice

$$\frac{1}{1-(t+t^2)} = \ ?$$

© Ministère de l'Éducation nationale

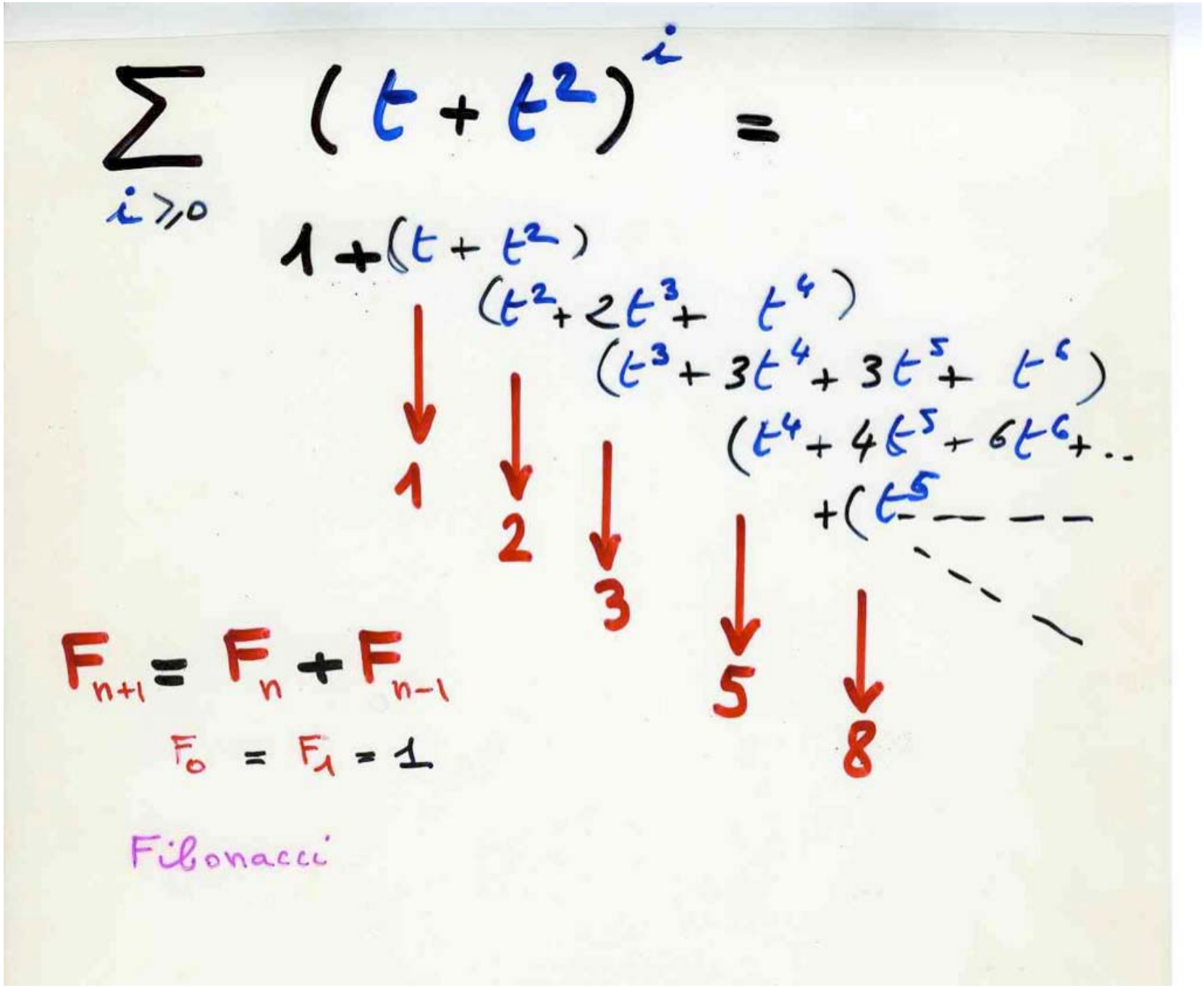


© Ministère de l'Éducation nationale

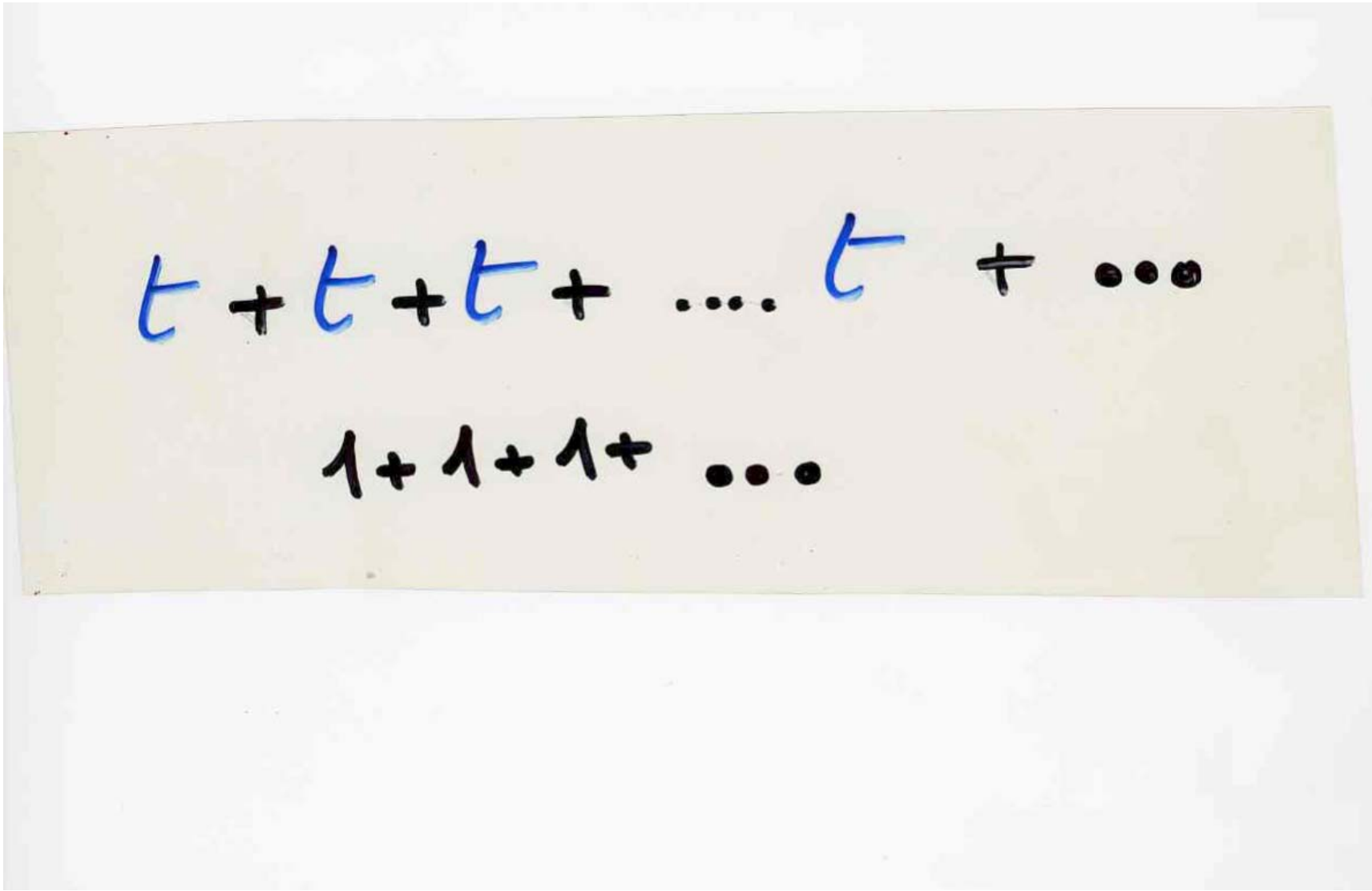
$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$
$$(t^2 + 2t^3 + t^4)$$
$$(t^3 + 3t^4 + 3t^5 + t^6)$$
$$(t^4 + 4t^5 + 6t^6 + \ldots$$
$$+ (t^5 - - - - -$$
$$\ddots$$

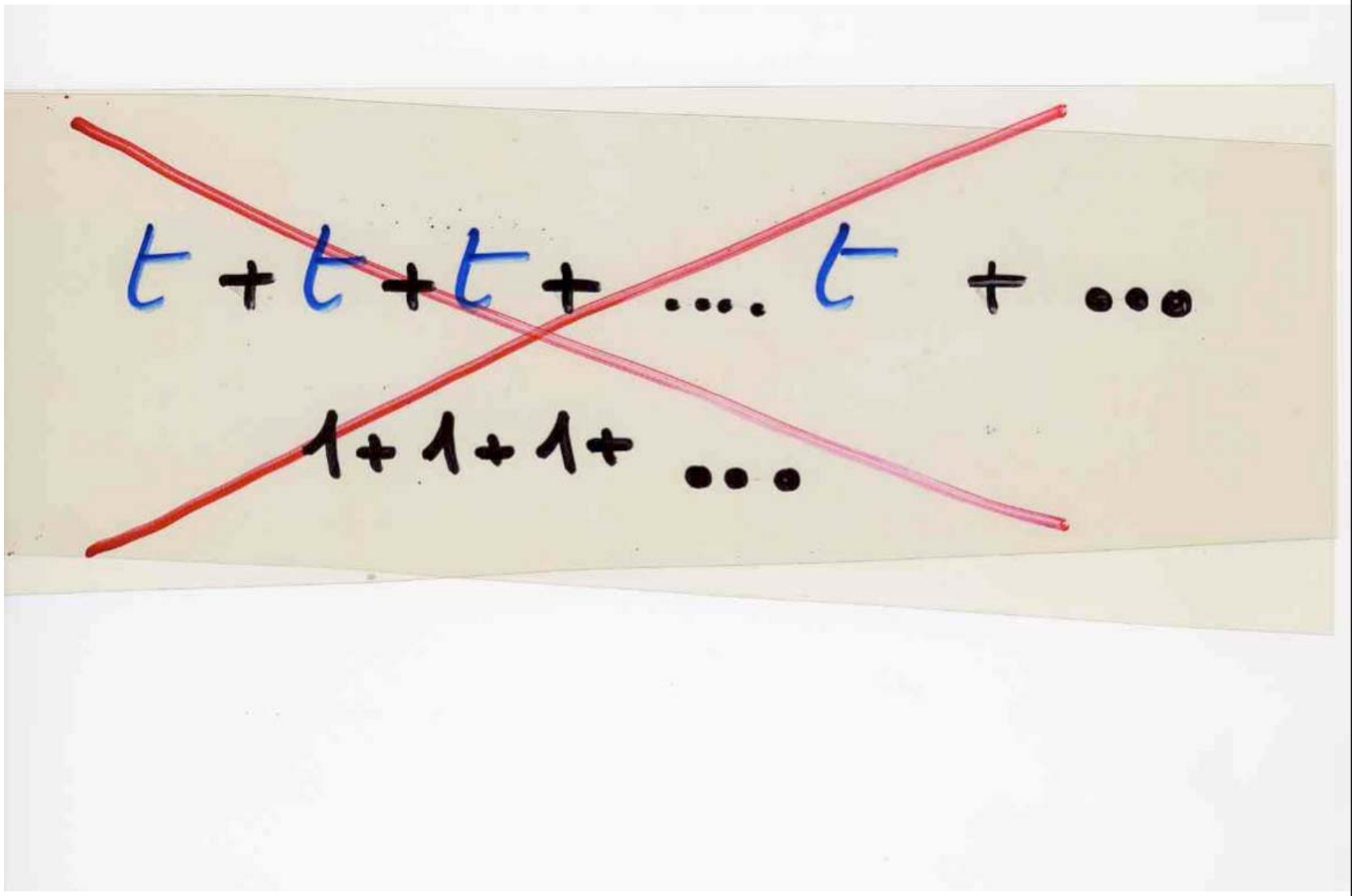
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

arbres binaires

© Ministère de l'Éducation nationale

recurrence

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

$$C_0 = 1$$

$$y = 1 + t y^2$$

equation algébrique

© Ministère de l'Éducation nationale

$$y = \frac{1 - (1 - 4t)^{1/2}}{2t}$$

© Ministère de l'Éducation nationale

$$(1+u)^m =$$

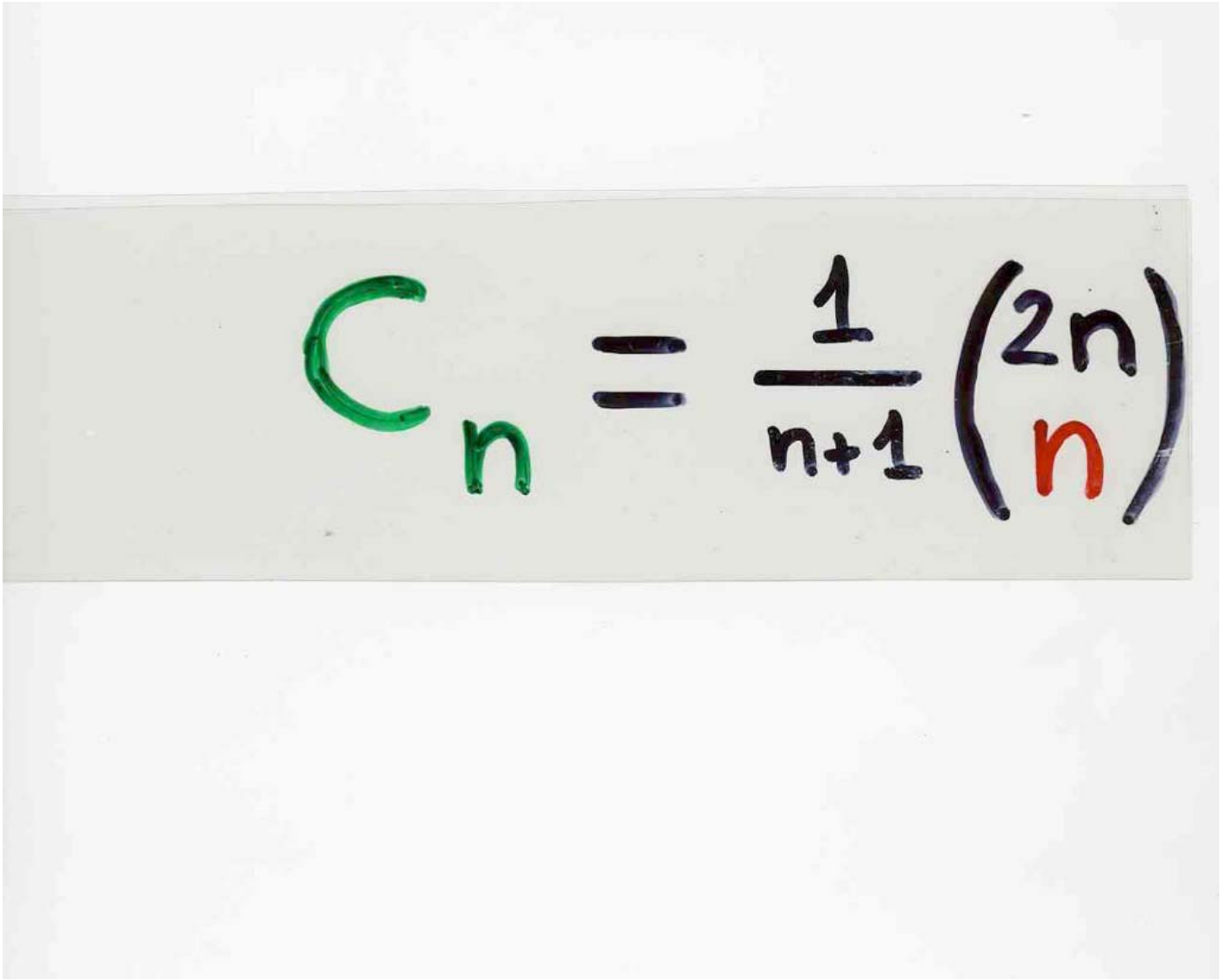
$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^2 + \frac{m(m-1)(m-2)}{3!}u^3 +$$

$$+ \ldots$$

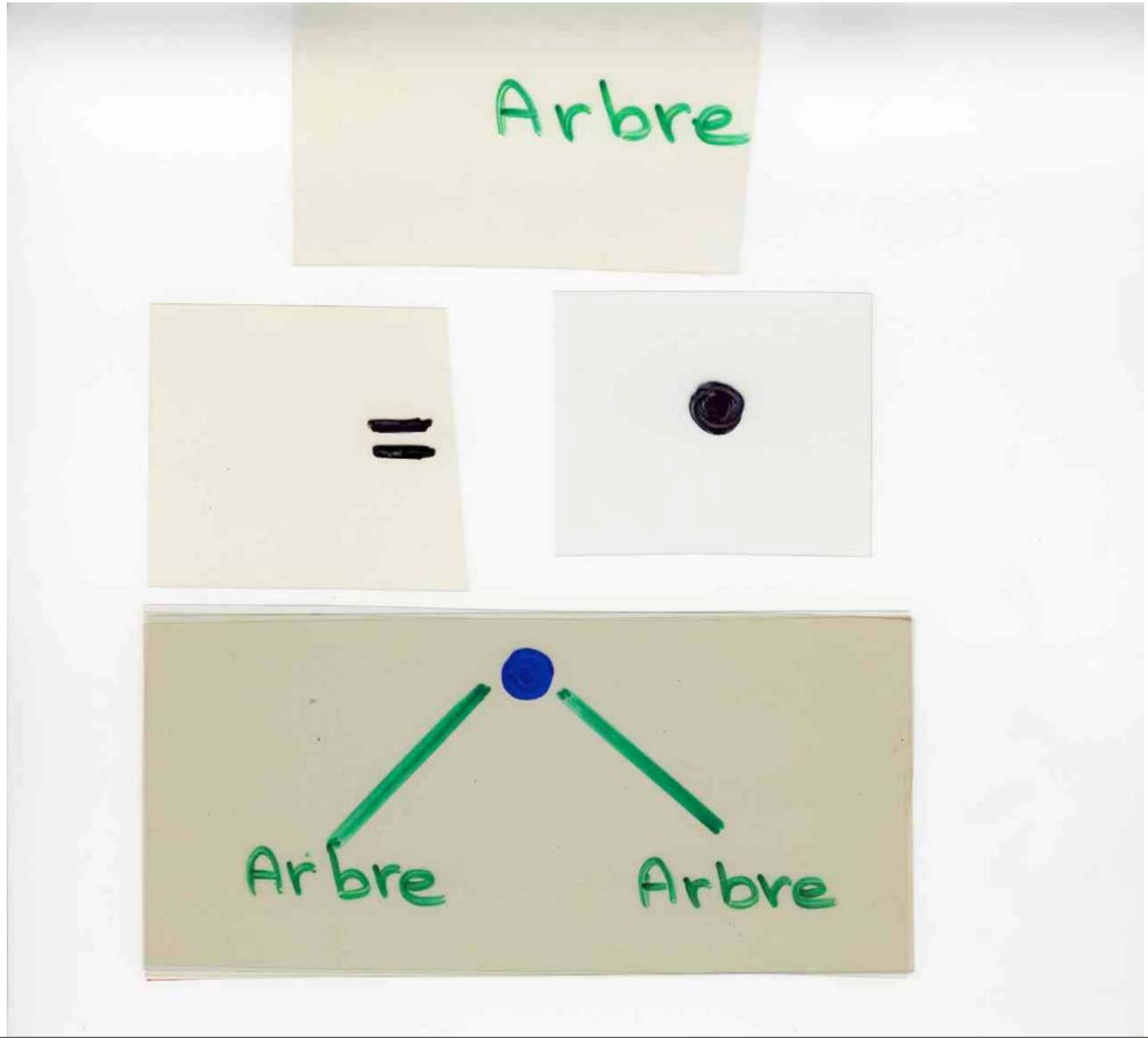
$$m = \frac{1}{2}$$

$$u = -4t$$

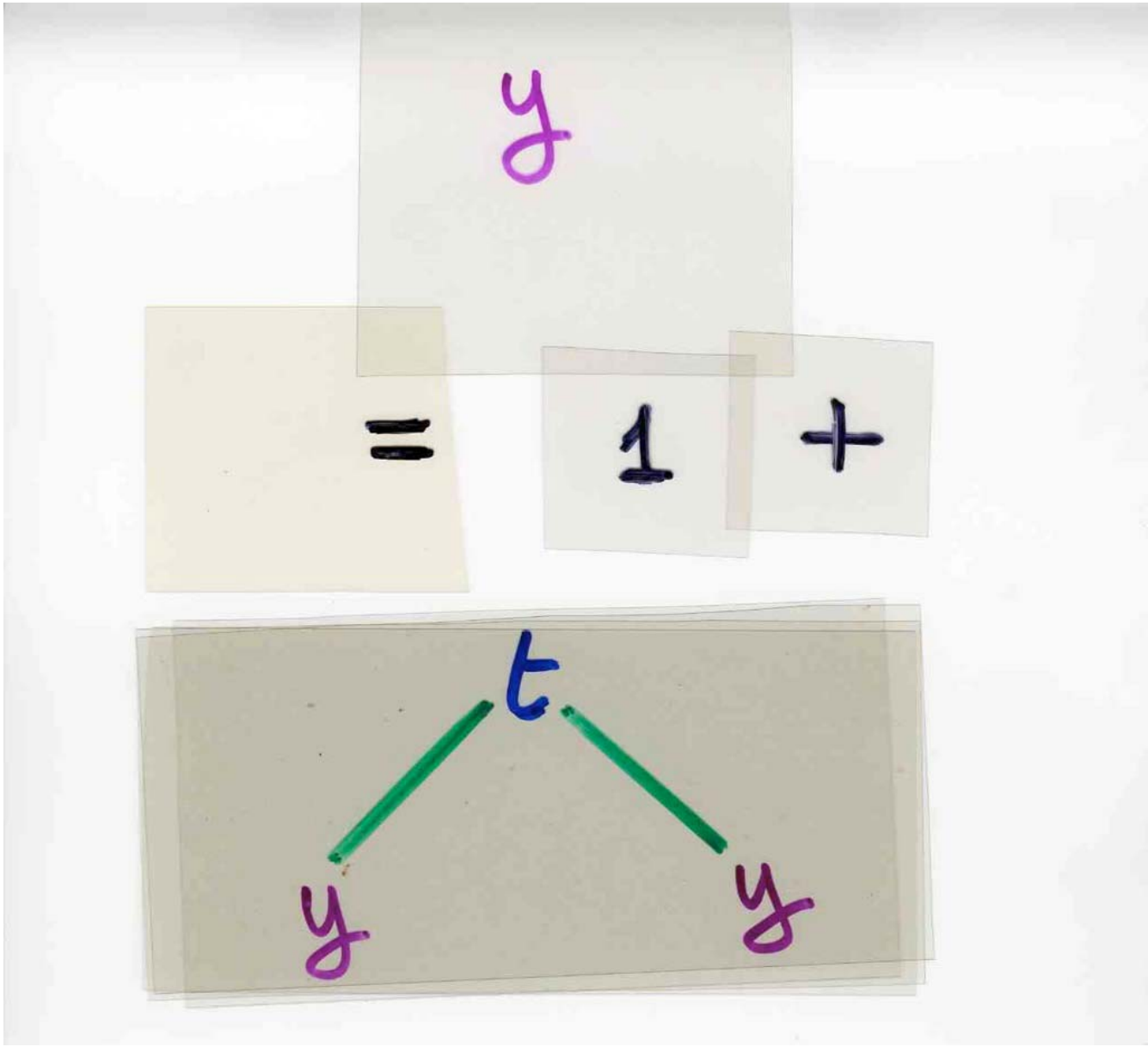
© Ministère de l'Éducation nationale



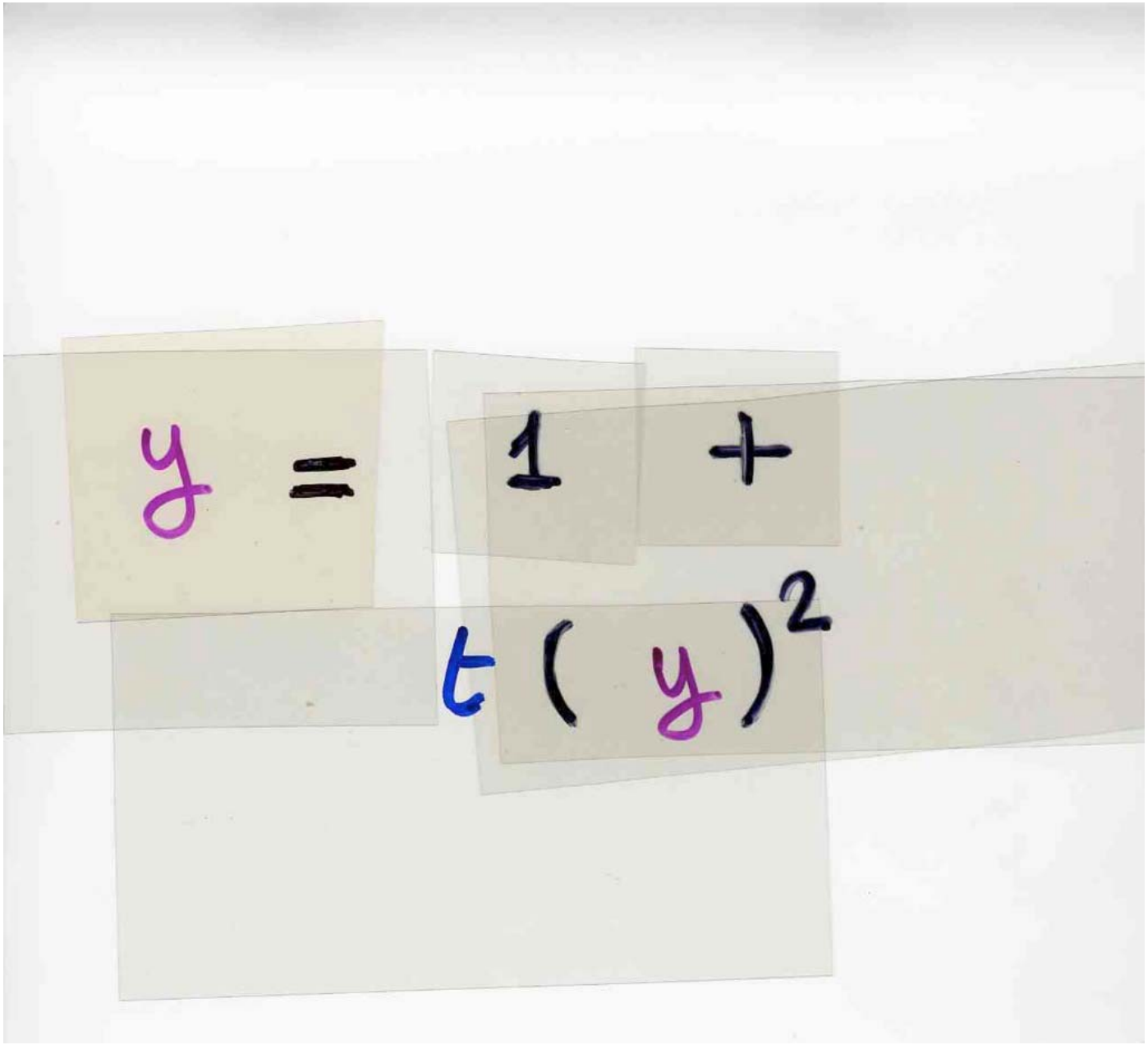
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

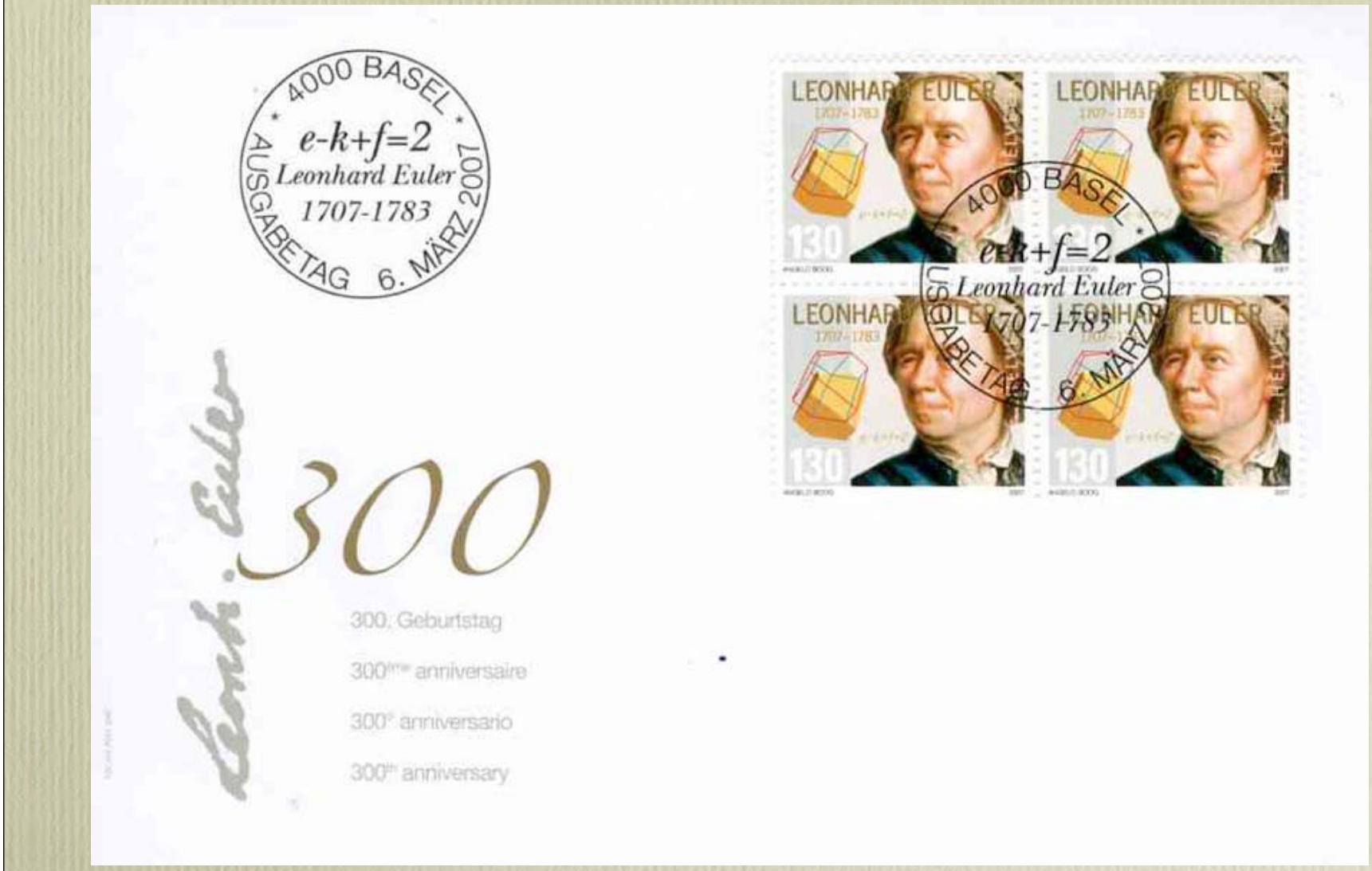


© Ministère de l'Éducation nationale

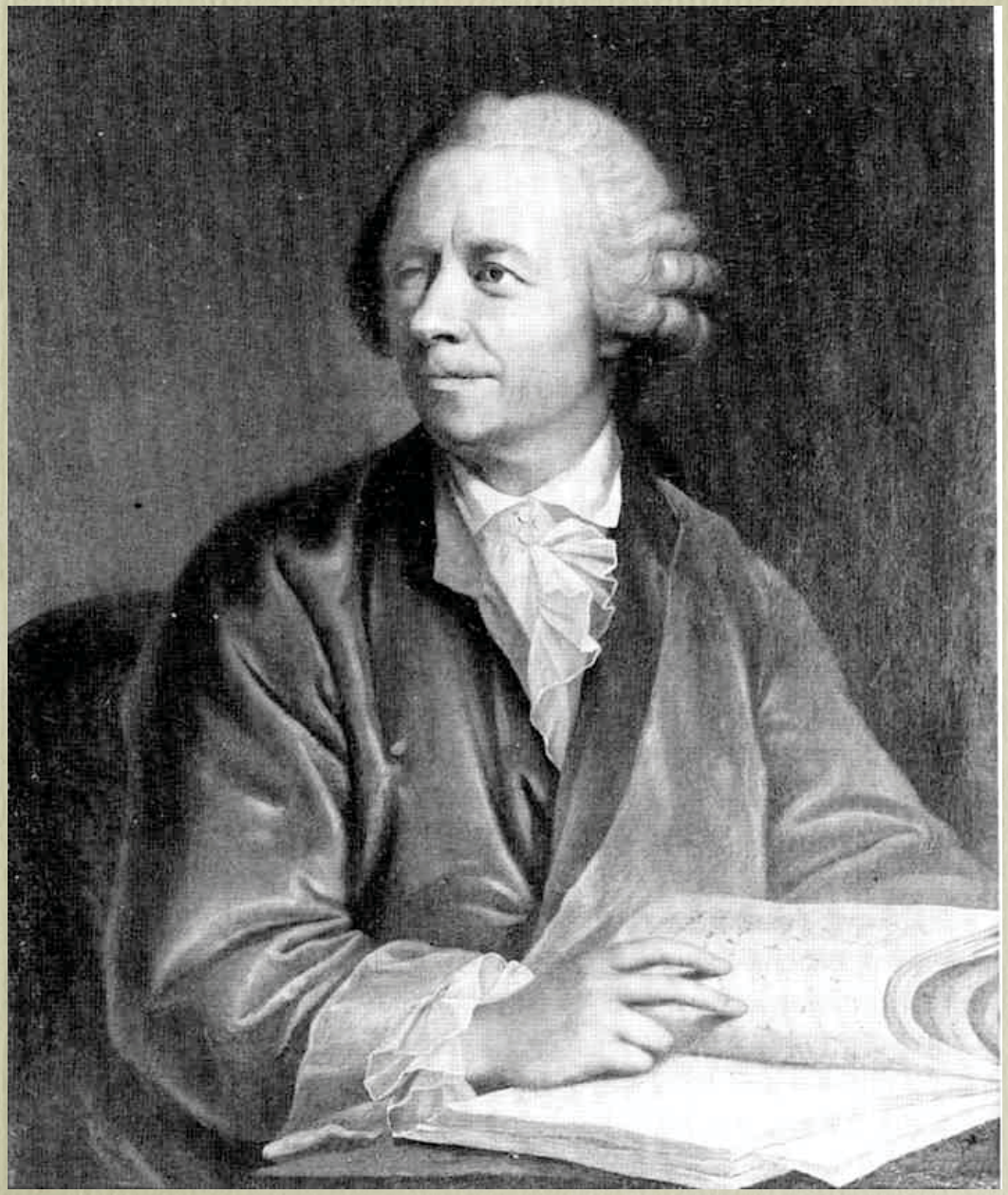
Une lettre oubliée d'Euler à Christian Goldbach ....

Berlin, 4 Septembre 1751

© Ministère de l'Éducation nationale

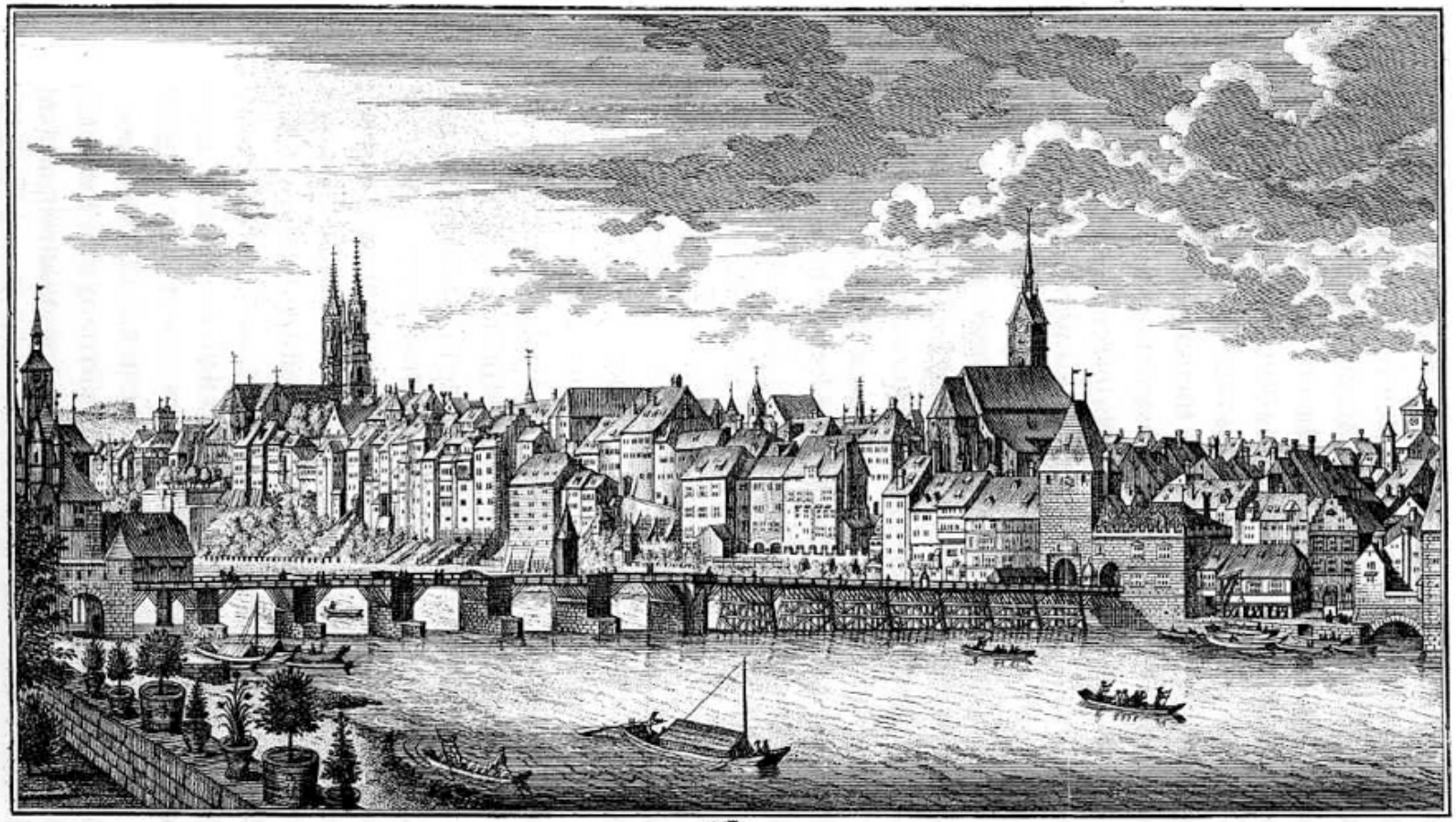


© Ministère de l'Éducation nationale



Leonhard
Euler
49 ans

© Ministère de l'Éducation nationale



PROSPECT DER RHEINBRÜCKE ZU BASEL
VON SEITEN DER KLEINEN STADT.
Em. Büchel del. 1761.

VUE DU PONT DU RHIN DE BASLE
DU CÔTÉ DE LA PETITE VILLE.
D. Herrliberger exc. Cum Priv.

© Ministère de l'Éducation nationale

Johann I Bernoulli
(1667 - 1748)

© Ministère de l'Éducation nationale

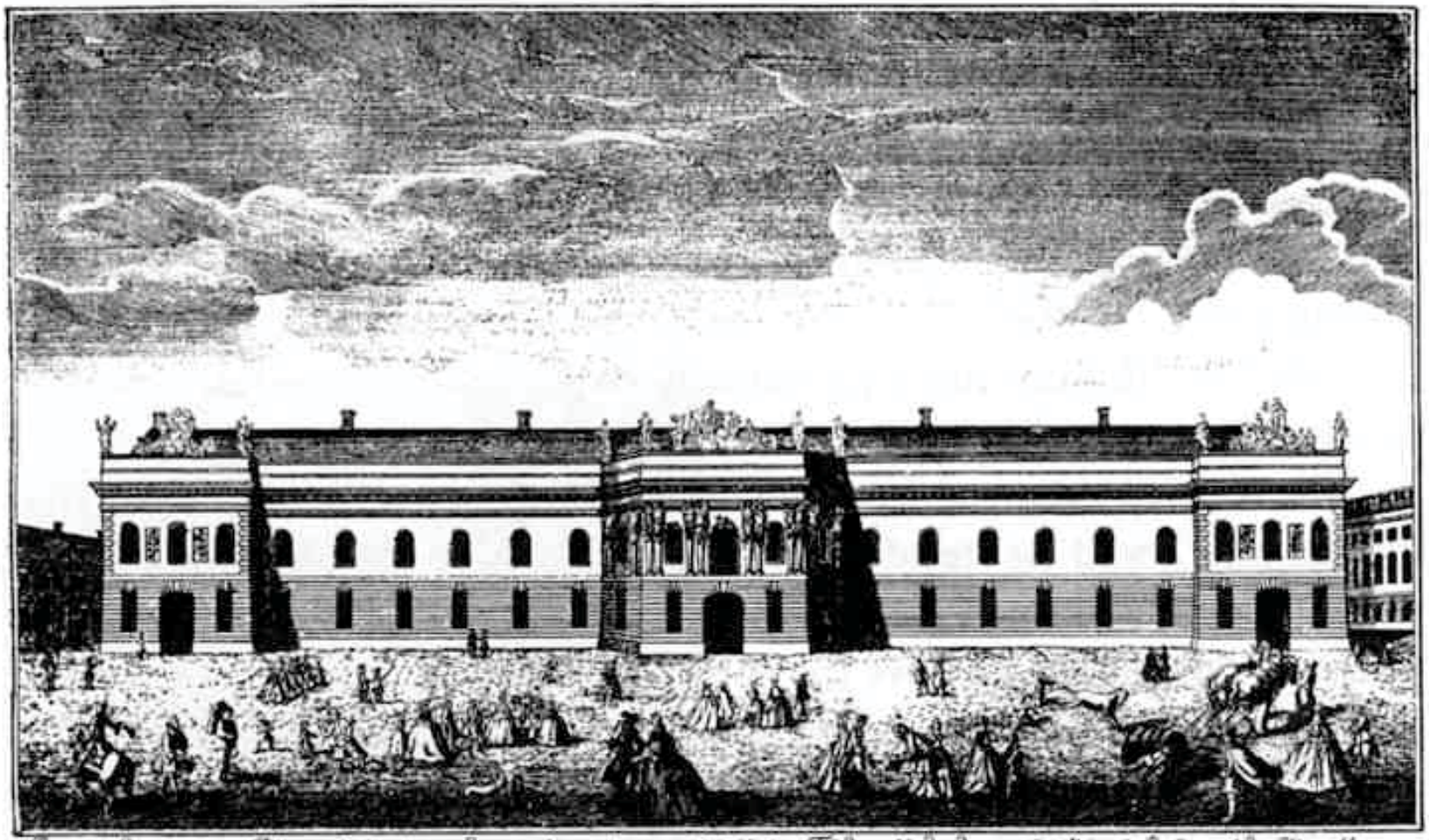
académie Saint-Pétersbourg

© Ministère de l'Éducation nationale

Niklaus II Bernoulli
(1695 - 1726)

Daniel I Bernoulli
(1700 - 1782)

© Ministère de l'Éducation nationale



Prospect der vor einigen Jahren abgebrannten, und nunmehro gantz neu aufgeführten Fördergebäudes des grossen Königl. Stalls auf der Dorotheen-Stadt zu Berlin, welches überaus ansehnliche Gebäude für die Königl. Academie der Wissenschaften und freyen Künste, und für die Academie der Künste und [illegible] Wissenschaften bestimmet ist.

123. **Die Akademie der Wissenschaften.** 1752

Nach einem Stiche von Schleuen

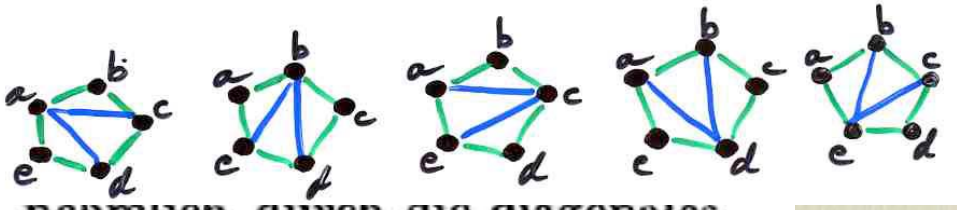
académie Berlin

© Ministère de l'Éducation nationale

Ich bin neulich auf eine Betrachtung gefallen, welche mir nicht wenig merkwürdig vorkam. Dieselbe betrifft, auf wie vielerley Arten ein gegebenes polygonum durch Diagonallinien in triangula zerschnitten werden könne.

Also ein quadrilaterum *abcd* kann entweder durch die diagonalem *ac*, oder durch *bd*, und also auf zweyerley Art in zwey triangula resolvirt werden.

Ein Fünfeck *abcde* wi



drey triangula getheilet, un

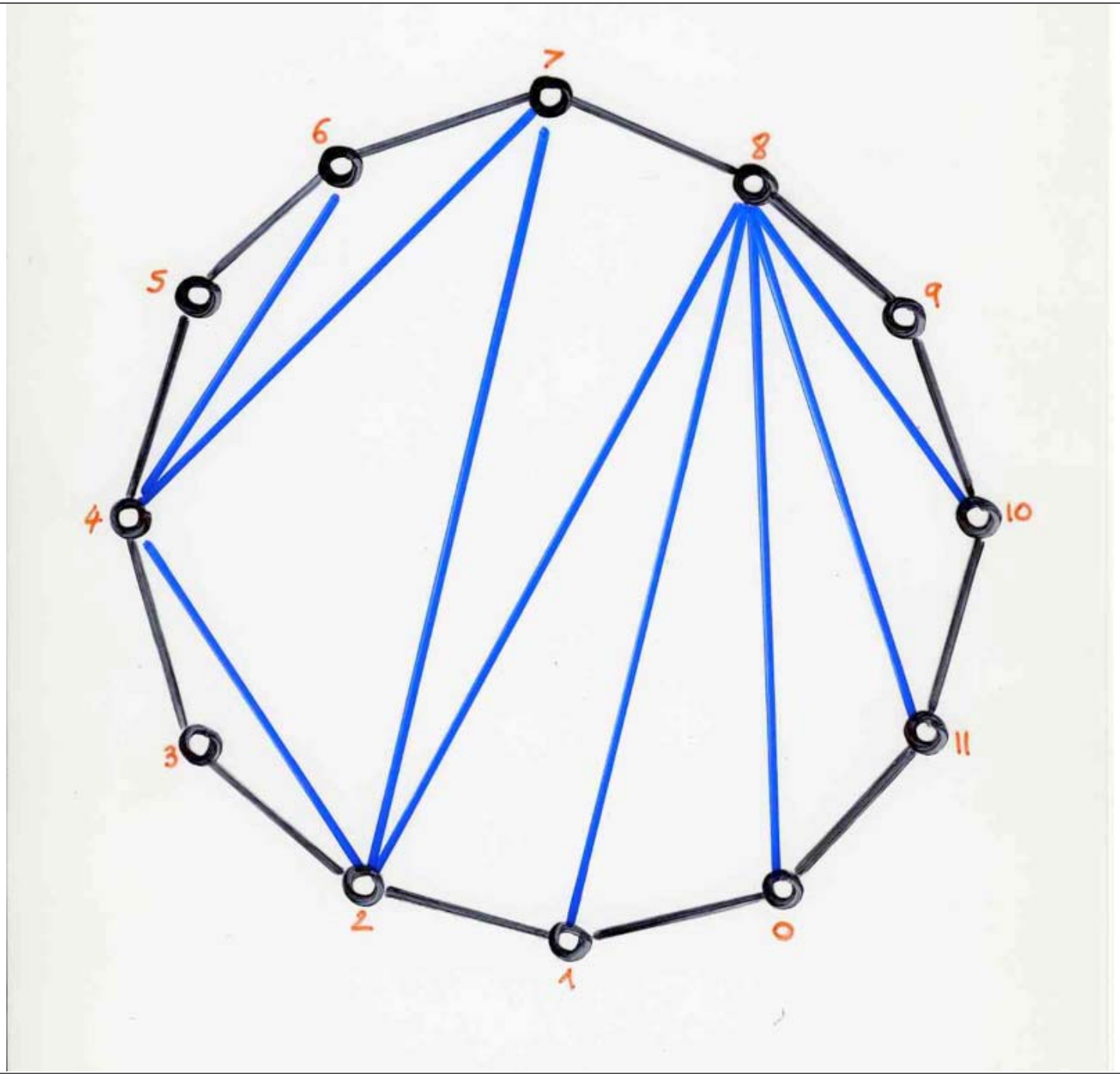
verschiedene Arten geschehen, nehmlich durch die diagonales

I. *ac*, *ad*. II. *bd*, *be*. III. *ca*; *ce*. IV. *db*, *da*, V. *ec*, *eb*.

Ferner wird ein Sechseck durch drey diagonales in vier triangula zertheilet, und dieses kann auf 14 verschiedene Arten geschehen.

Nun ist die Frage generaliter: da ein polygonum von *n* Seiten durch $n-3$ diagonales in $n-2$ triangula zerschnit-

Direction générale de l'Enseignement scolaire
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

ten wird, auf wie vielerley verschiedene Arten solches geschehen könne. Setze ich nun die Anzahl dieser verschiedenen Arten $= x$, so habe ich per inductionem gefunden

wenn $n = 3, 4, 5, 6, 7, 8, 9, 10$

so ist $x = 1, 2, 5, 14, 42, 132, 429, 1430.$

Hieraus habe ich nun den Schluss gemacht, dass generaliter sey

$C_n = \frac{1}{n+1}\binom{2n}{n}$

$$x = \frac{2.6.10.14.18.22....(4n-10)}{2.3.4.5.6.7.....(n-1)}$$

ist $1 = \frac{2}{2}$, $2 = 1.\frac{6}{3}$, $5 = 2.\frac{10}{4}$, $14 = 5.\frac{14}{5}$, $42 = 14.\frac{18}{6}$, $132 = 42.\frac{22}{7}$; dass also aus einer jeden Zahl die folgende leicht gefunden wird. Die Induction aber, so ich gebraucht, war ziemlich mühsam, doch zweifle ich nicht, dass diese Sach nicht sollte weit leichter entwickelt werden können. Ueber die Progression der Zahlen 1, 2, 5, 14, 42, 132, etc. habe ich auch diese Eigenschaft angemerket, dass $1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$. Also wenn $a = \frac{1}{4}$, so ist

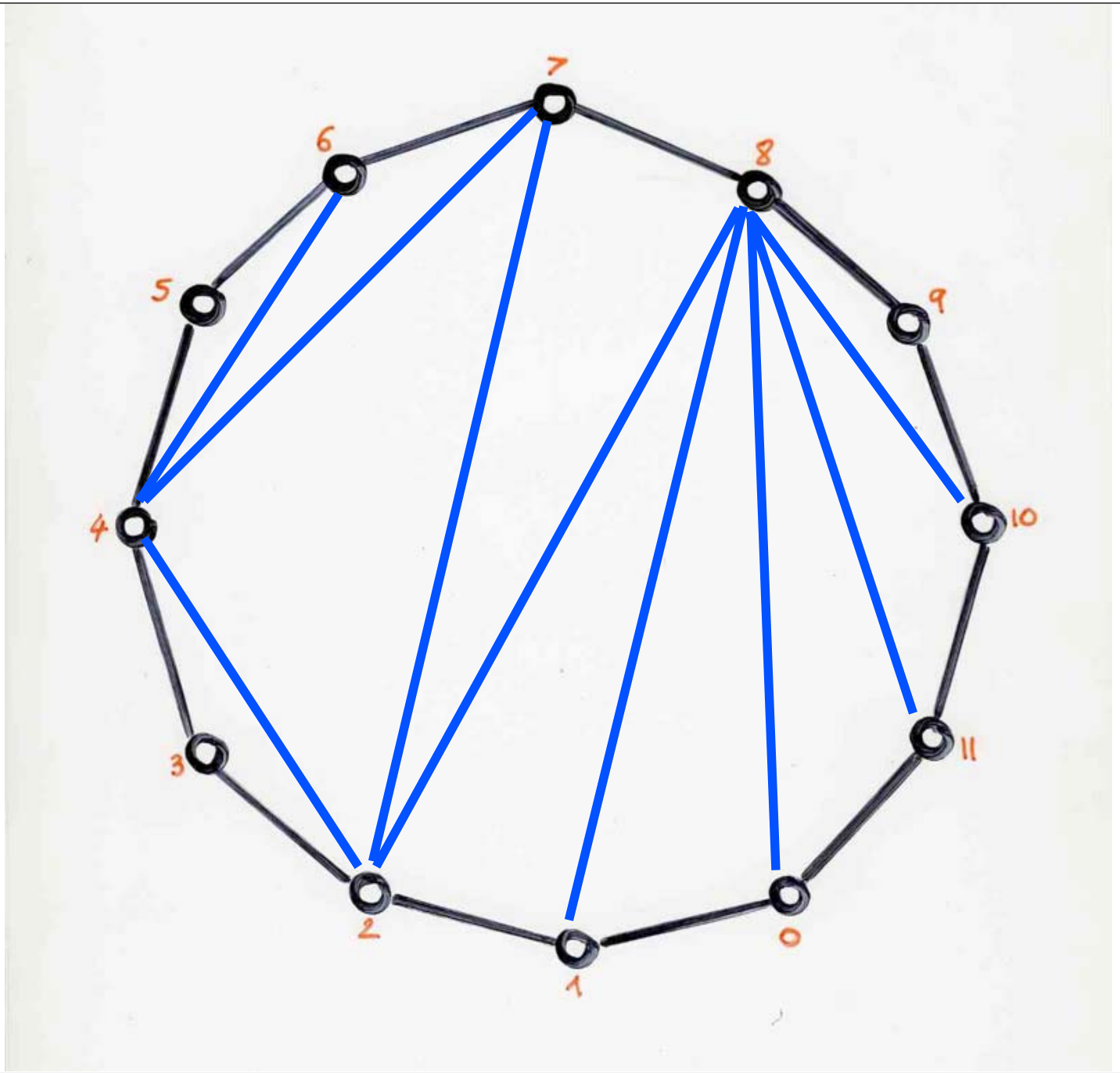
$$1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4.$$

Euler.

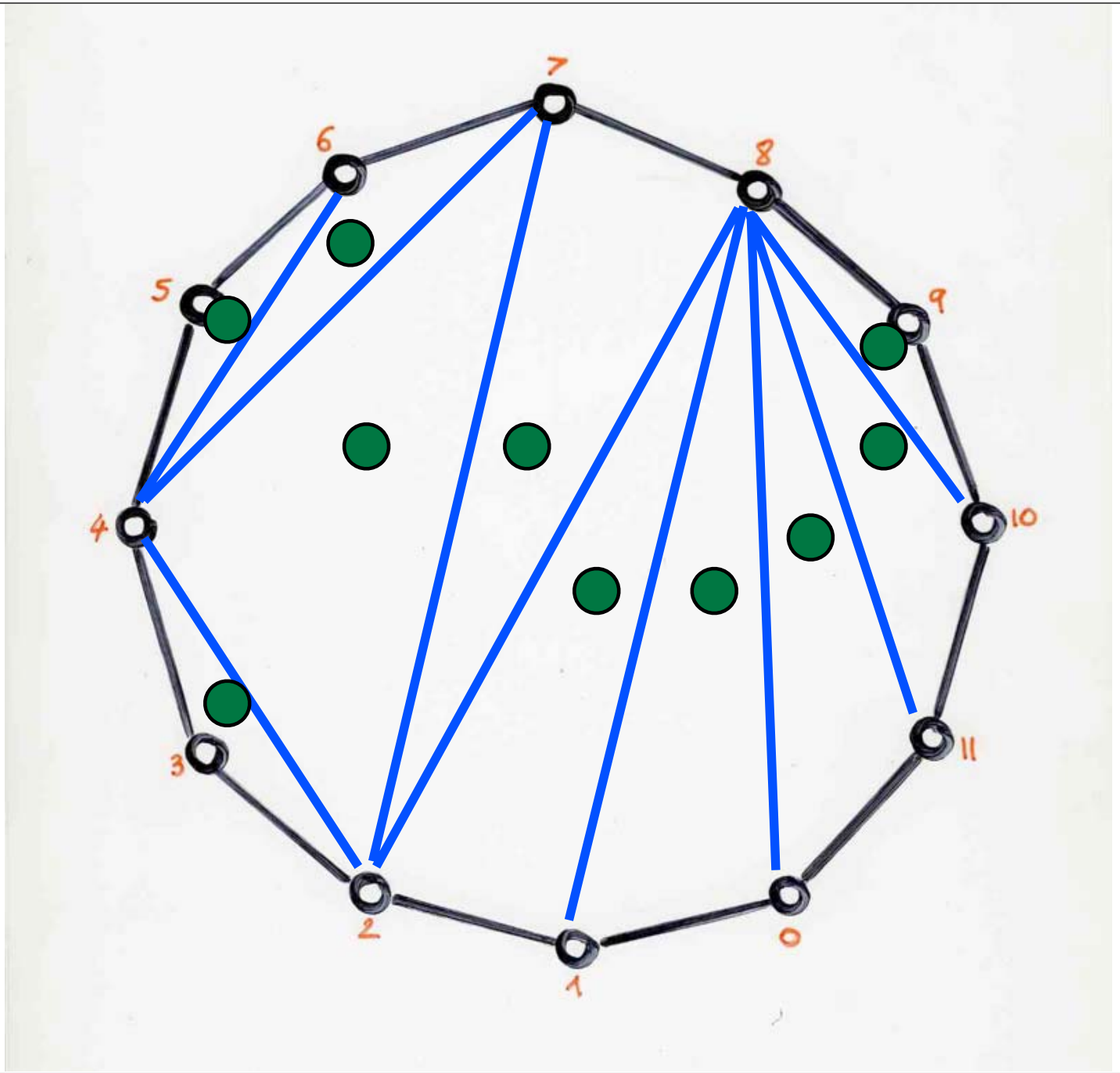
© Ministère de l'Éducation nationale

des triangulations aux arbres binaires

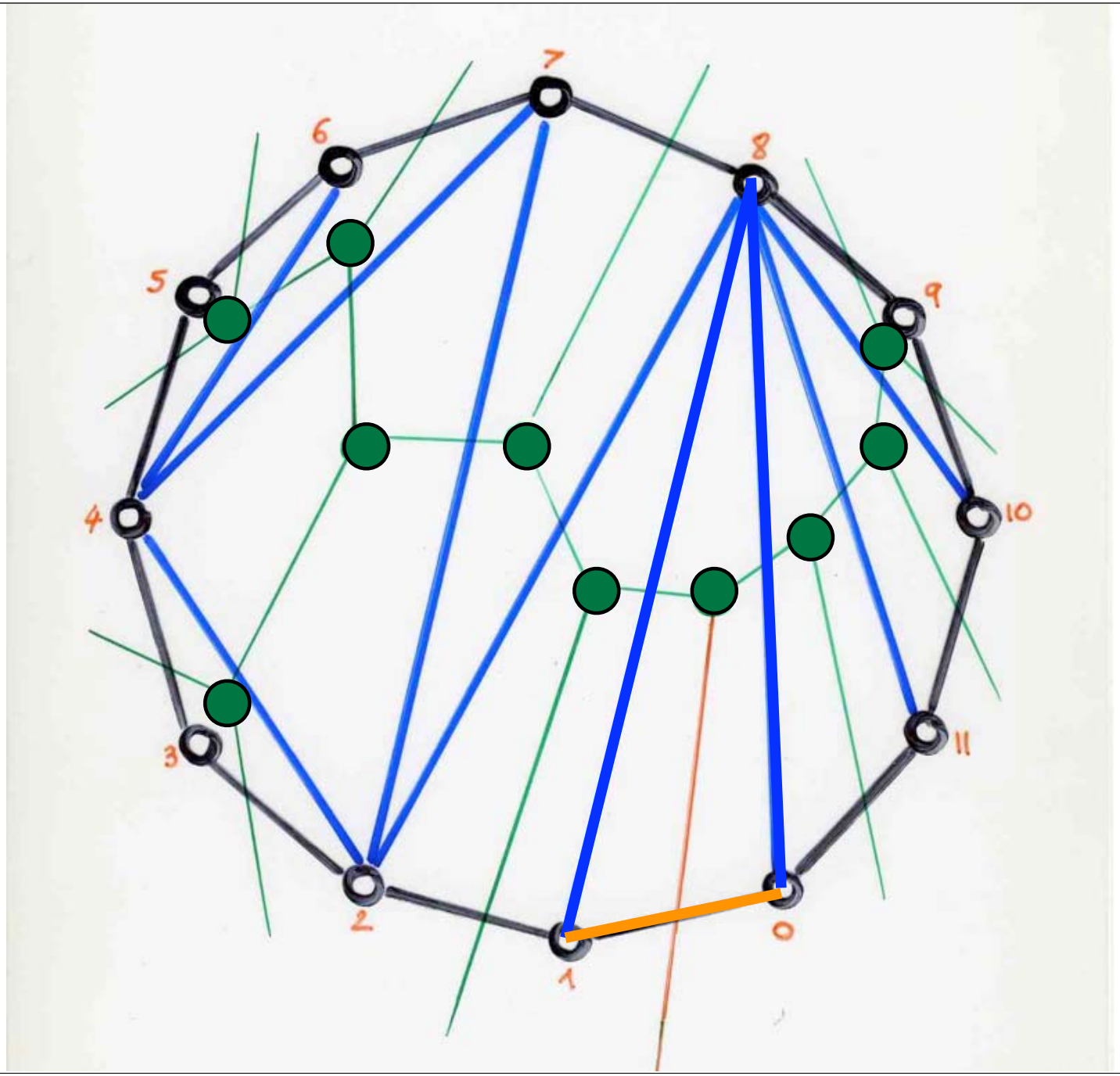
© Ministère de l'Éducation nationale



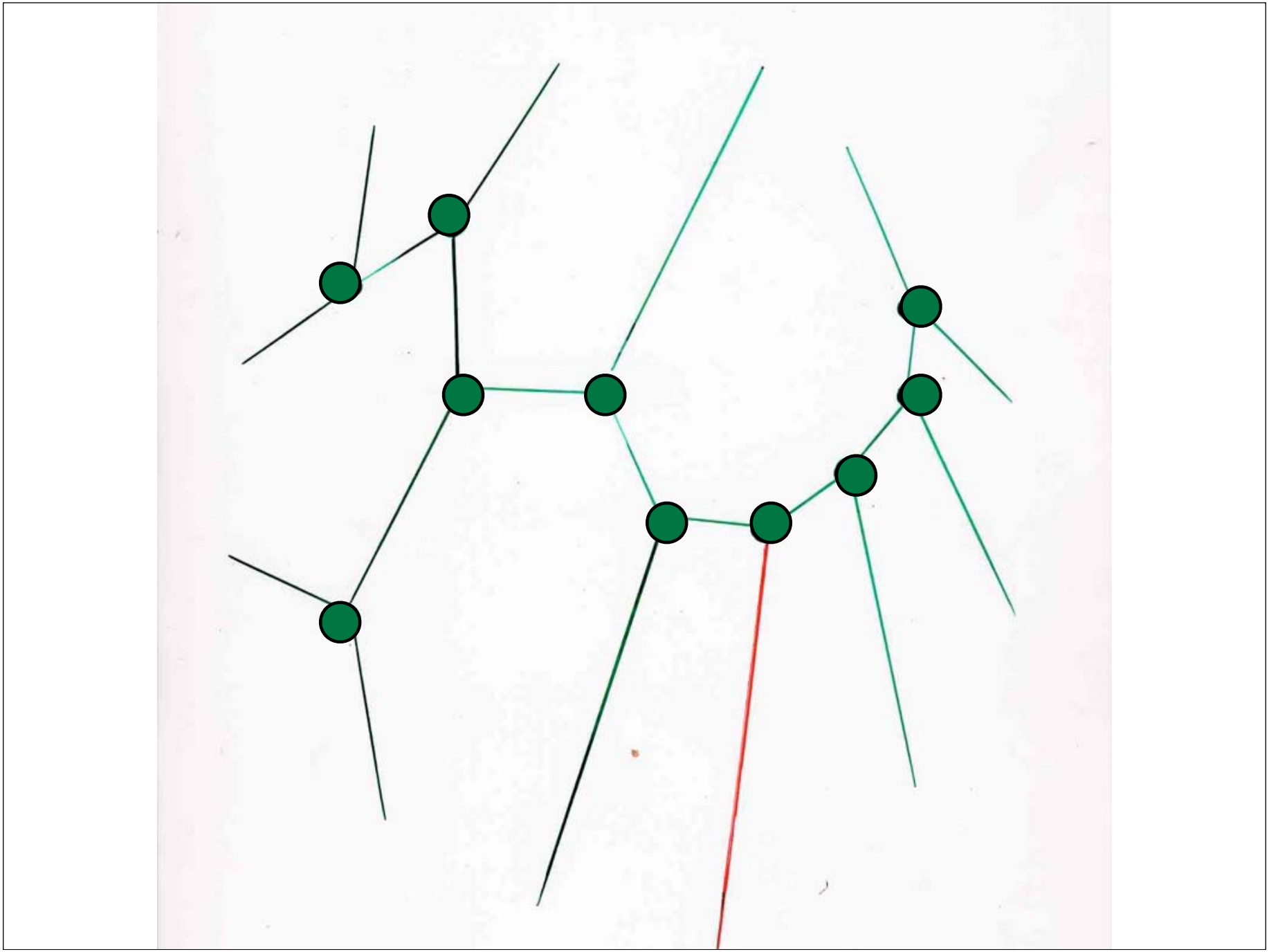
© Ministère de l'Éducation nationale



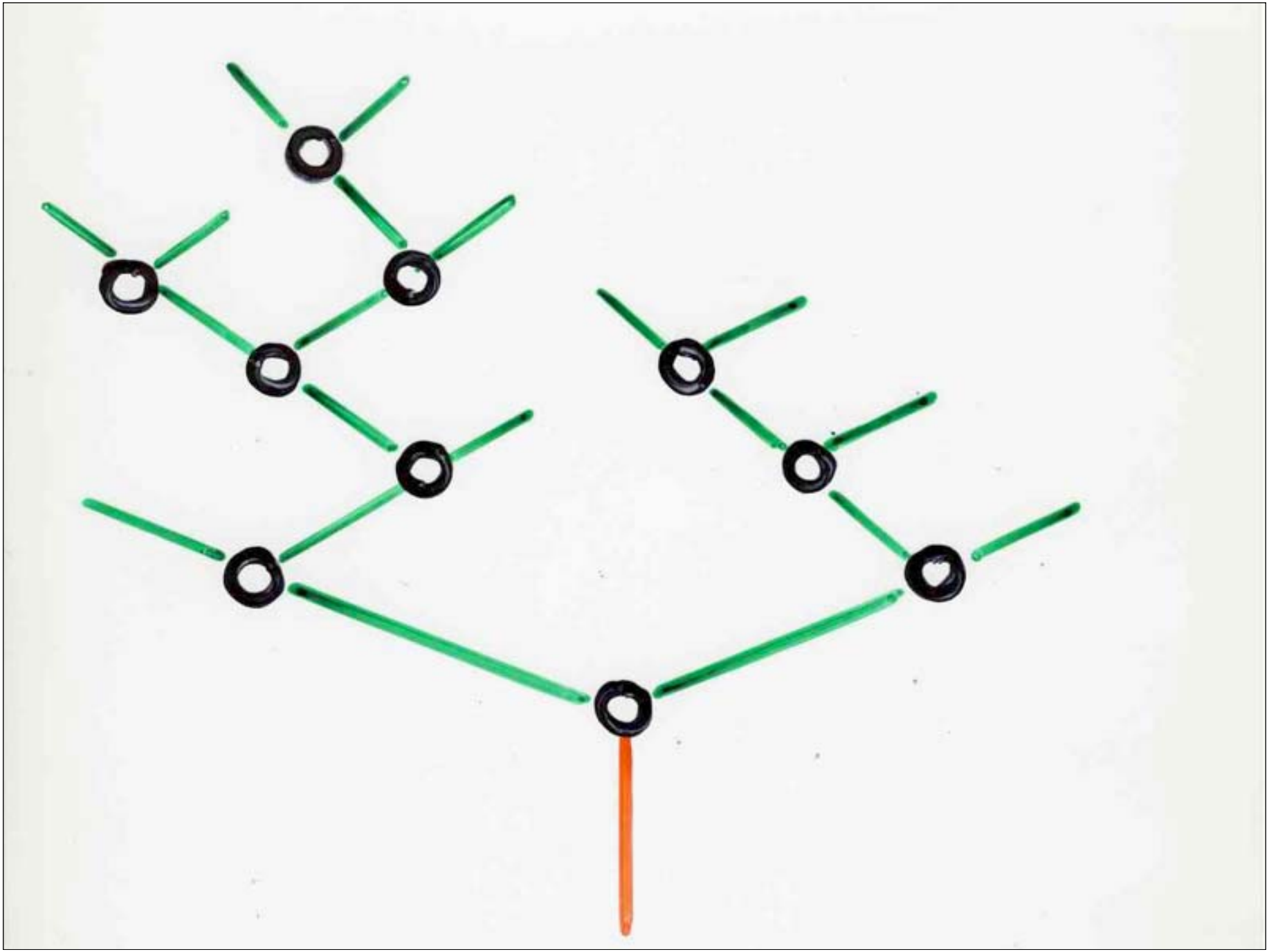
© Ministère de l'Éducation nationale



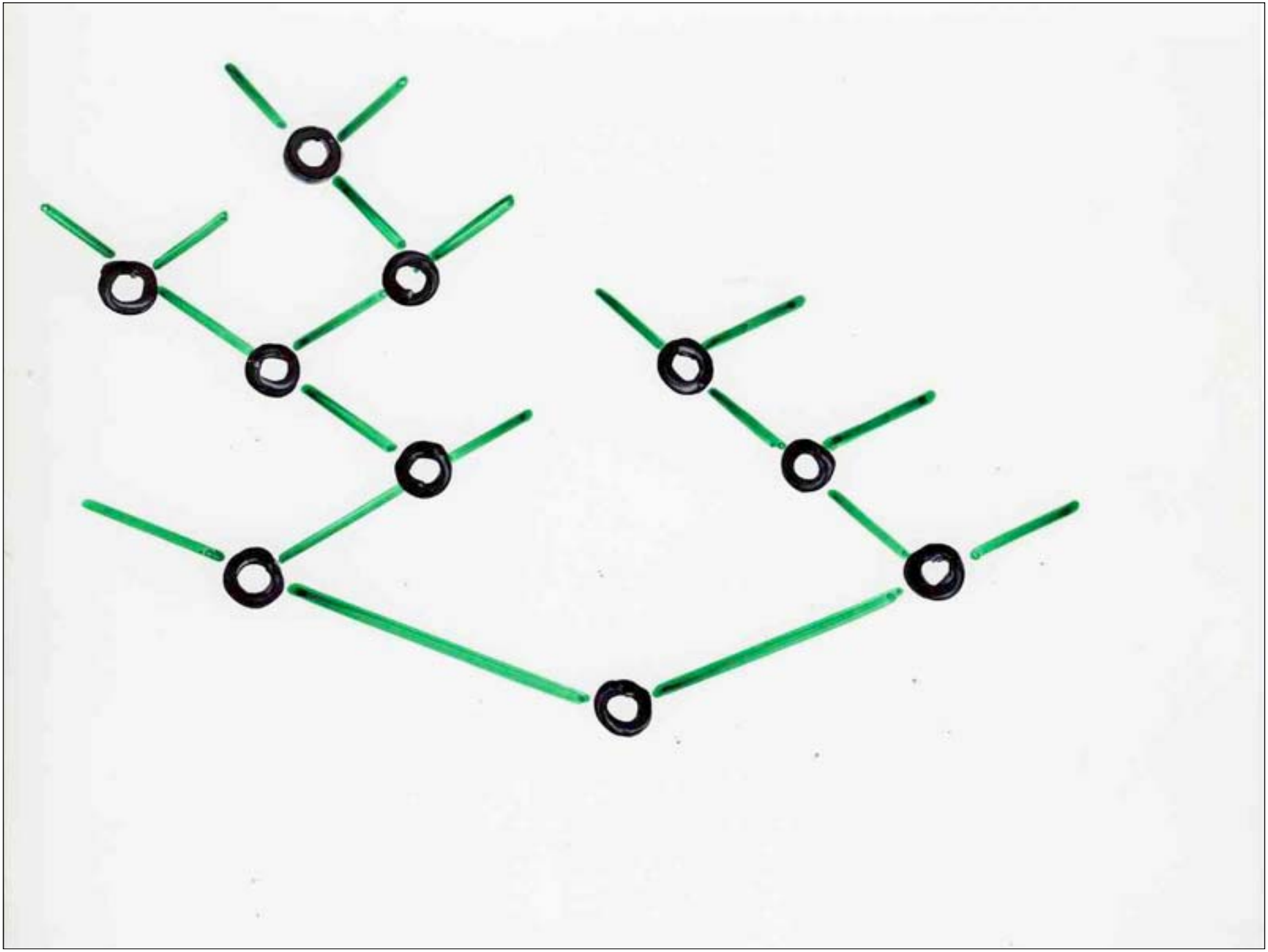
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

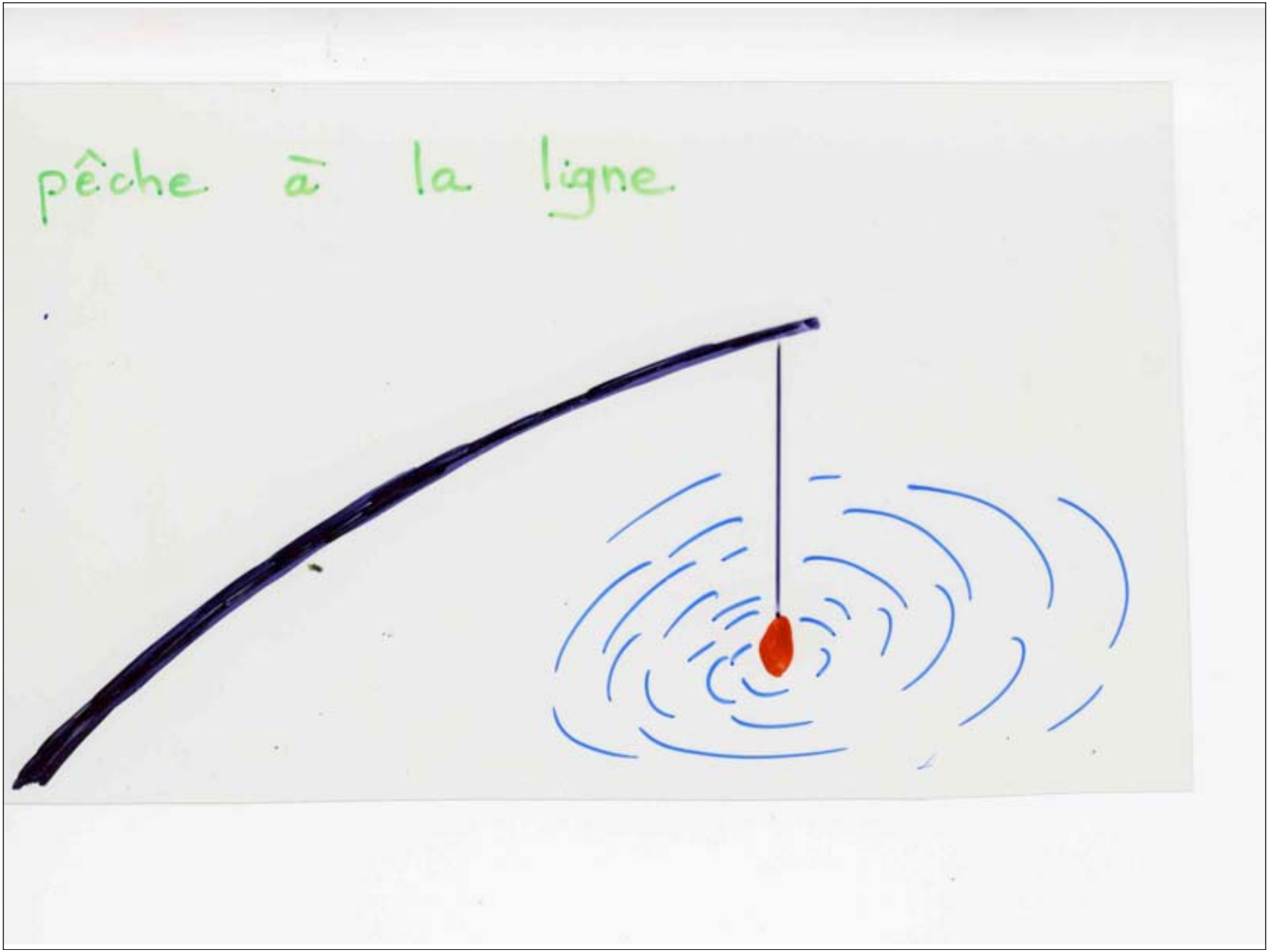


© Ministère de l'Éducation nationale

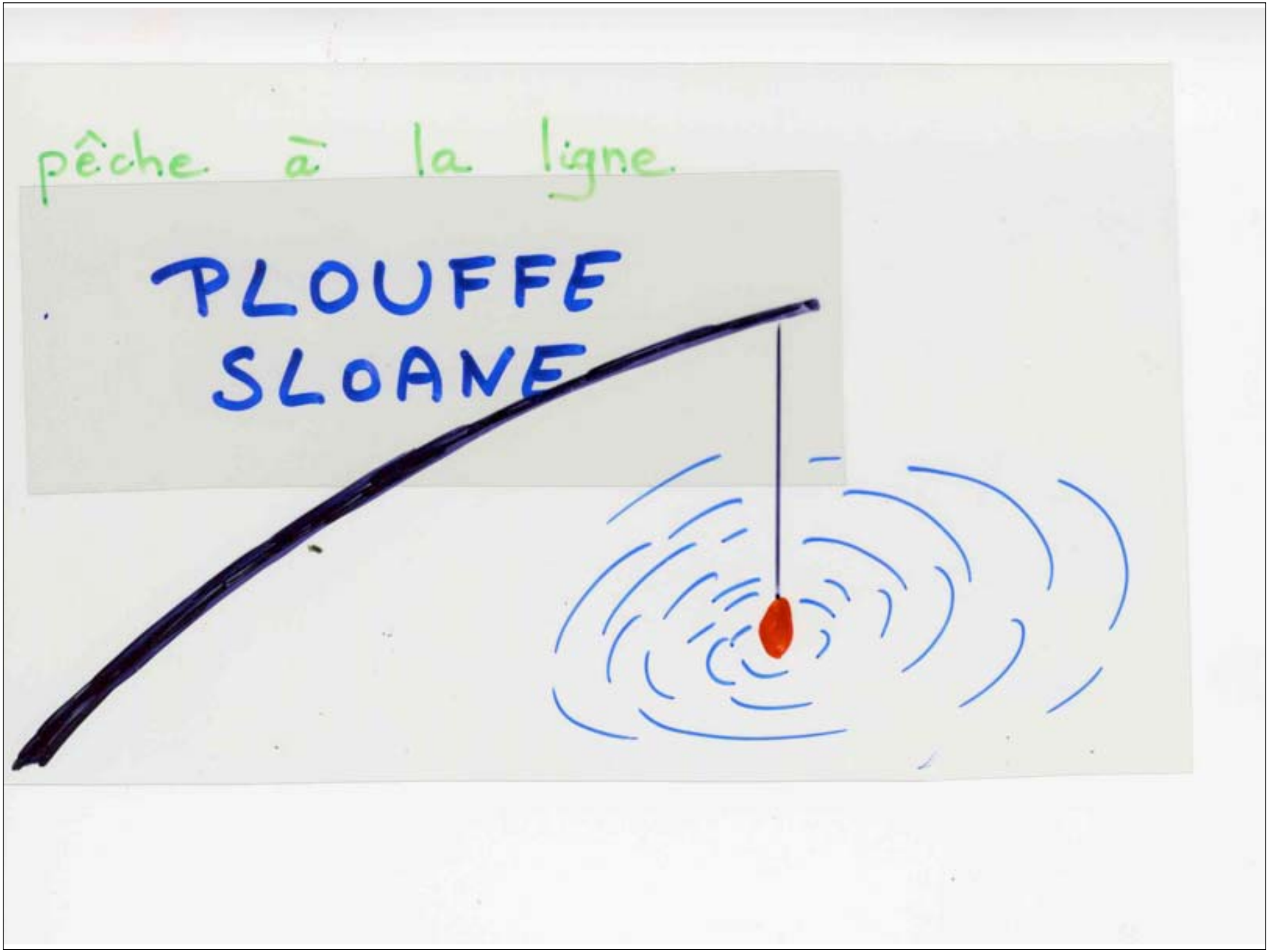
combinatoire expérimentale

avec ....

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

THE ENCYCLOPEDIA OF INTEGER SEQUENCES

N. J. A. Sloane
Mathematical Sciences Research Center
AT&T Bell Laboratories
Murray Hill, New Jersey

Simon Plouffe
Département de Mathématiques et d'Informatique
Université du Québec à Montréal
Montréal, Québec

ACADEMIC PRESS
San Diego New York Boston London Sydney Tokyo Toronto

Direction générale de l'Enseignement scolaire
© Ministère de l'Éducation nationale

M2819 1, 3, 9, 35, 178
van der Waerden numbers. Ref Loth83 49. [1,2; A5346]

M2820 1, 3, 9, 35, 201, 1827
Coefficients of Bell's formula. Ref NMT 10 65 62. [2,2; A2575, N1134]

M2821 1, 3, 9, 37, 153, 951, 5473, 42729, 353937, 3455083, 30071001, 426685293, 4707929449, 59350096287, 882391484913, 15177204356401, 205119866263713
Sums of logarithmic numbers. Ref TMS 31 79 63. jos. [0,2; A2751, N1135]

M2822 1, 1, 1, 3, 9, 37, 177, 959, 6097, 41641, 325249, 2693691, 24807321, 241586893, 2558036145, 28607094455, 342232522657, 4315903789009, 57569080467073
Expansion of $e^{\tan x}$. Ref JO61 150. [0,4; A6229]

M2823 1, 3, 9, 42, 206, 1352, 10168
Regular semigroups of order n. Ref PL65. MAL 2 2 67. SGF 14 71 77. [1,2; A1427, N1136]

M2824 0, 1, 1, 3, 9, 45, 225, 1575, 11025, 99225, 893025, 9823275, 108056025, 1404728325, 18261468225, 273922023375, 4108830350625, 69850115960625
Expansion of $1 / (1-x)(1-x^2)^{1/2}$. Ref R1 87. [1,4; A0246, N1137]

M2825 1, 1, 1, 3, 9, 48, 504, 14188, 1351563
Threshold functions of n variables. Ref PGEC 19 821 70. MU71 38. [0,4; A1530, N1138]

M2826 3, 9, 54, 450, 4725, 59535, 873180, 14594580
Expansion of an integral. Ref C1 167. [2,1; A1194, N1139]

M2827 1, 3, 9, 89, 1705, 67774
Superpositions of cycles. Ref AMA 131 143 73. [3,2; A3225]

M2828 1, 3, 9, 93, 315, 3855, 13797, 182361, 9256395, 34636833, 1857283155, 26817356775, 102280151421, 1497207322929, 84973577874915, 488526061274087
Fermat quotients: $(2^{p-1}-1)/p$. Ref Well86 70. [0,2; A7663]

M2829 3, 10, 4, 5, 10, 2, 5, 3, 2, 3, 6, 6, 6, 3, 5, 6, 10, 5, 5, 10, 6, 6, 6, 2, 5, 8, 2, 6, 8, 4, 4, 6, 4, 5, 10, 2, 4, 7, 11, 5, 7, 9, 10, 7, 1, 6, 7, 11, 7, 10, 0, 6, 8, 9, 6, 4, 7, 13, 2, 6, 4, 32,
Iterations until $3n$ reaches 153 under x goes to sum of cubes of digi[illegible] Robe[illegible]415040
[1,1; A3620]

M2830 1, 3, 10, 13, 62, 75, 437, 512, 949, 6206, 13361, 7301[illegible]
6193523, 26639450, 59472423, 383473988, 1593368375, [illegible]
Convergents to cube root of 3. Ref AMP 46 105 1866. L1 67[illegible]

© Ministère de l'Éducation nationale

M1453 1, 2, 5, 14, 36, 98, 273, 768, 2197, 6360, 18584, 54780, 162672, 486154, 1461197, 4413988, 13393855, 40807290, 124783669, 382842018, 1178140280, 3635626680
Secondary alcohols with n carbon atoms. Ref BA76 44. [3,2; A5955]

M1454 2, 5, 14, 38, 107
Domino n-tuples. Ref JRM 7 324 74. [1,1; A6574]

M1455 1, 1, 2, 5, 14, 38, 120, 353, 1148, 3527, 11622, 36627, 121622, 389560, 1301140, 4215748, 13976335, 46235800, 155741571, 512559185, 1732007938, 5732533570
Folding a strip of n blank stamps. See Fig M4587. Ref ScAm 209(3) 262 63. JCT 5 151 68. CBUL (2) 3 36 75. [1,3; A1011, N0576]

M1456 1, 2, 5, 14, 39, 109
Paraffins with n carbon atoms. Ref ZFK 93 437 36. [1,2; A0641, N0575]

M1457 1, 1, 2, 5, 14, 40, 128, 369, 1214, 3516, 12776, 40534, 137404, 463232, 1602348, 5216253, 17753898, 58597316, 212150928, 710453534, 2366853608, 8584498376
Shifts left under l.c.m.-convolution with itself. Ref BeSl94. [0,3; A7463]

M1458 1, 2, 5, 14, 41, 122, 365, 1094, 3281, 9842, 29525, 88574, 265721, 797162, 2391485, 7174454, 21523361, 64570082, 193710245, 581130734, 1743392201
$(3^n+1)/2$. Ref BPNR 60. Ribe91 53. HM94. [0,2; A7051]

M1459 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020
Catalan numbers: $C(n) = C(2n,n)/(n+1)$. See Fig M1459. Ref AMM 72 973 65. RCI 101. C1 53. PLC 2 109 71. MAG 61 211 88. [0,3; A0108, N0577]

$$\text{G.f.: } \frac{1-(1-4x)^{1/2}}{2x}.$$

Figure M1459. CATALAN NUMBERS.

The **Catalan numbers**, defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}C(2n,n)$$

(M1459), are probably the most frequently occurring combinatorial numbers after the binomial coefficients. [GO4] lists over 240 references. See also [MAG 45 199 61], [AMM 72 973 65], [PLC 2 109 71], [C1], [Stan86] and [GKP]. Some of the dozens of interpretations of C_n are:

(i) The number of ways of dissecting a convex polygon of $n+2$ sides into n triangles by drawing $n-1$ nonintersecting diagonals (Fig. (a)).

(ii) The number of ways of completely parenthesizing a product of $n+1$ letters (so that there are two factors inside each set of parentheses):

$n = 1$ (ab); $n = 2$ $a(bc)$, $(ab)c$;
$n = 3$ $(ab)(cd)$, $a((bc)d)$, $((ab)c)d$, $a(b(cd))$, $(a(bc))d$.

© Ministère de l'Éducation nationale

Domino n-tuples. Ref JRM 7 324 74. [1,1; A0574]

M1455 1, 1, 2, 5, 14, 38, 120, 353, 1148, 3527, 11622, 36627, 121622, 389560, 1301140, 4215748, 13976335, 46235800, 155741571, 512559185, 1732007938, 5732533570
Folding a strip of n blank stamps. See Fig M4587. Ref ScAm 209(3) 262 63. JCT 5 151 68. CBUL (2) 3 36 75. [1,3; A1011, N0576]

M1456 1, 2, 5, 14, 39, 109
Paraffins with n carbon atoms. Ref ZFK 93 437 36. [1,2; A0641, N0575]

M1457 1, 1, 2, 5, 14, 40, 128, 369, 1214, 3516, 12776, 40534, 137404, 463232, 1602348, 5216253, 17753898, 58597316, 212150928, 710453534, 2366853608, 8584498376
Shifts left under l.c.m.-convolution with itself. Ref BeSl94. [0,3; A7463]

M1458 1, 2, 5, 14, 41, 122, 365, 1094, 3281, 9842, 29525, 88574, 265721, 797162, 2391485, 7174454, 21523361, 64570082, 193710245, 581130734, 1743392201
$(3^n + 1)/2$. Ref BPNR 60. Ribe91 53. HM94. [0,2; A7051]

M1459 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020
Catalan numbers: $C(n) = C(2n,n)/(n+1)$. See Fig M1459. Ref AMM 72 973 65. RCI 101. C1 53. PLC 2 109 71. MAG 61 211 88. [0,3; A0108, N0577]

$$\text{G.f.: } \frac{1 - (1 - 4x)^{1/2}}{2x}.$$

© Ministère de l'Éducation nationale

merorum exponit, ac rigoroſe demonſtrat, dum docet, quomodo pro quouis polygono reſolutionum numerus, ex cognita reſolutione polygonorum ſimpliciorum, quae paucioribus conſtant lateribus, colligi debeat. Hac ratione, ſi a ſimpliciſſimis incipiamus, hanc inueſtigationem continuo ad polygona plurium laterum extendere licet, ſicque Ill. Auctor ſub finem tabellam adiecit, in qua iſtiusmodi reſolutiones ad polygona 20 laterum vsque exhibet. Liceat autem nobis, a ſummo quodam Geometra, qui eandem hanc tabulam calculo ſubiecit, admonitis, obſeruare, hanc tabulam, ob quendam calculi errorem, tantum vsque ad polygona 15 laterum eſſe iuſtam, quippe pro hoc polygono reſolutionum numerus non eſt 751900, vt tabella habet, ſed 742900, ſequentes quoque numeri, dum forte nouus error irrepſit, primo ad 17 vsque latera nimis ſunt magni, deinde vero nimis parui, dum pro 20 lateribus reſolutionum numerus eſt 477638700. Facilius haec apparent, ſi ex lege primum obſeruata, qua quilibet numerus ex omnibus praecedentibus colligitur, alia ad computum facilior eliciatur, cuius ope quilibet numerus ex ſolo praecedente definiatur. Ita ſi pro polygono n laterum numerus reſolutionum ſit P, pro polygono ſequente $n+1$ laterum numerus reſolutionum erit $\frac{4n-6}{n}$ P. Quin etiam hinc, ſine conſideratione praecedentium, ſtatim indefinite pro polygono n laterum numerus reſolutionum ita per factores exprimitur, vt ſit:

$$\frac{4}{2}\cdot\frac{6}{3}\cdot\frac{10}{4}\cdot\frac{14}{5}\cdot\frac{18}{6}\cdot\frac{22}{7}\cdot\ldots\ldots\ldots\frac{4n-10}{n-1}$$

vbi numeratores quaternario, denominatores vero vnitate creſcunt. Hinc ſequentem nouam tabulam, benevole

vole nobiscum communicatam, adiungere e re viſum eſt, quod Ill. Auctori huius ſchediasmatis non diſpliciturum eſſe ſperamus.

Num. laterum	numerus reſolutionum.	num. laterum	numerus reſolutionum.
III	1	XV	742900
IV	2	XVI	2674440
V	5	XVII	9694845
VI	14	XVIII	35357670
VII	42	XIX	129644790
VIII	132	XX	477638700
IX	429	XXI	1767263190
X	1430	XXII	6564120420
XI	4862	XXIII	24466267020
XII	16796	XXIV	91482563640
XIII	58786	XXV	343059613650.
XIV	208012		

VII.

Methodus ſimplex et vniuerſalis omnes omnium aequationum radices detegendi.

Auct. I. A. de Segner pag. 211.

Complures iam a Geometris excogitatae ſunt methodi aequationum algebraicarum radices, vel accurate, vel proxime ſaltem, determinandi: omnes autem fere poſtulant, vt valores radicum, quae quae-

runtur,

Direction générale de l'Enseignement scolaire
© Ministère de l'Éducation nationale

NOVI COMMENTARII ACADEMIAE SCIENTIARVM IMPERIALIS PETROPOLITANAE

TOM. VII.

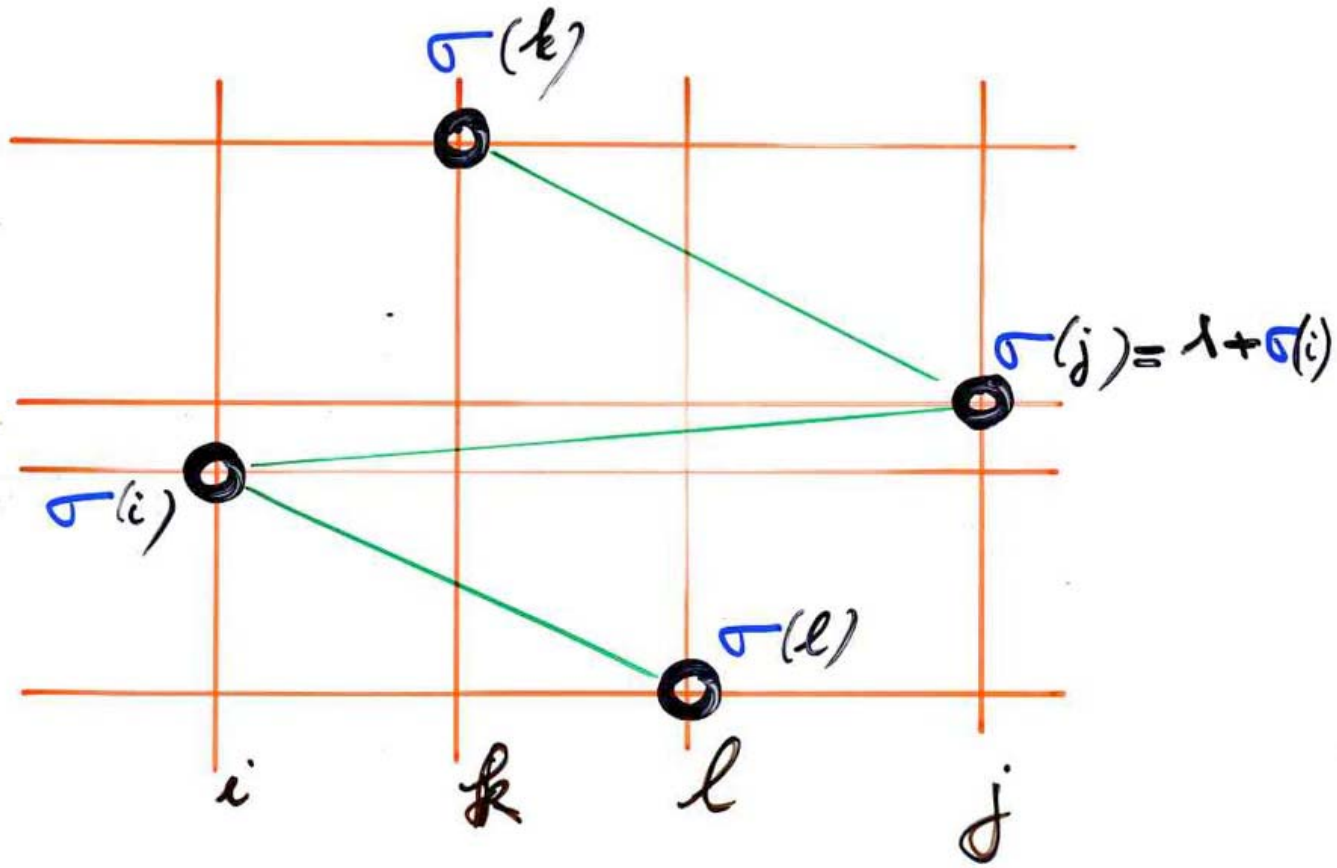
pro Annis MDCCLVIII. et MDCCLIX.

PETROPOLI
TYPIS ACADEMIAE SCIENTIARVM
MDCCLXI.

un exemple:

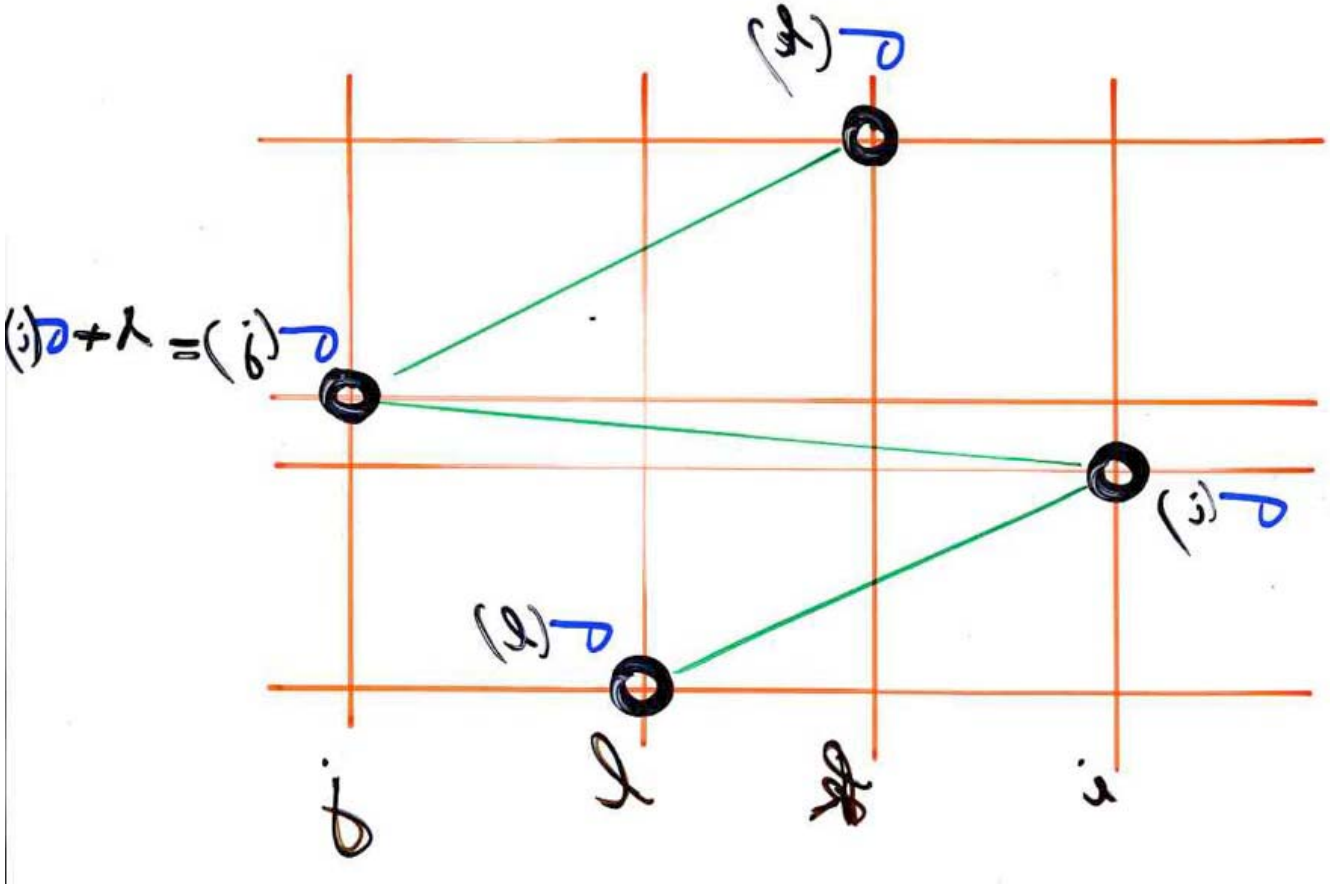
permutations de Baxter

© Ministère de l'Éducation nationale

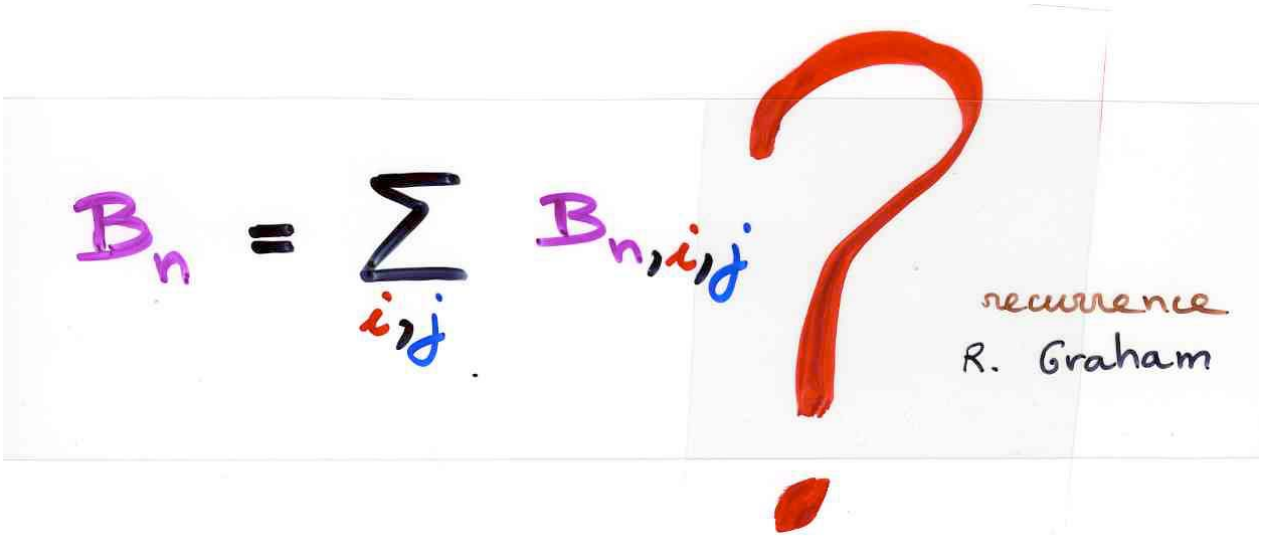


	1	2	3	4	5	6	7	8	9
σ =	5	1	2	4	3	9	7	8	6

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



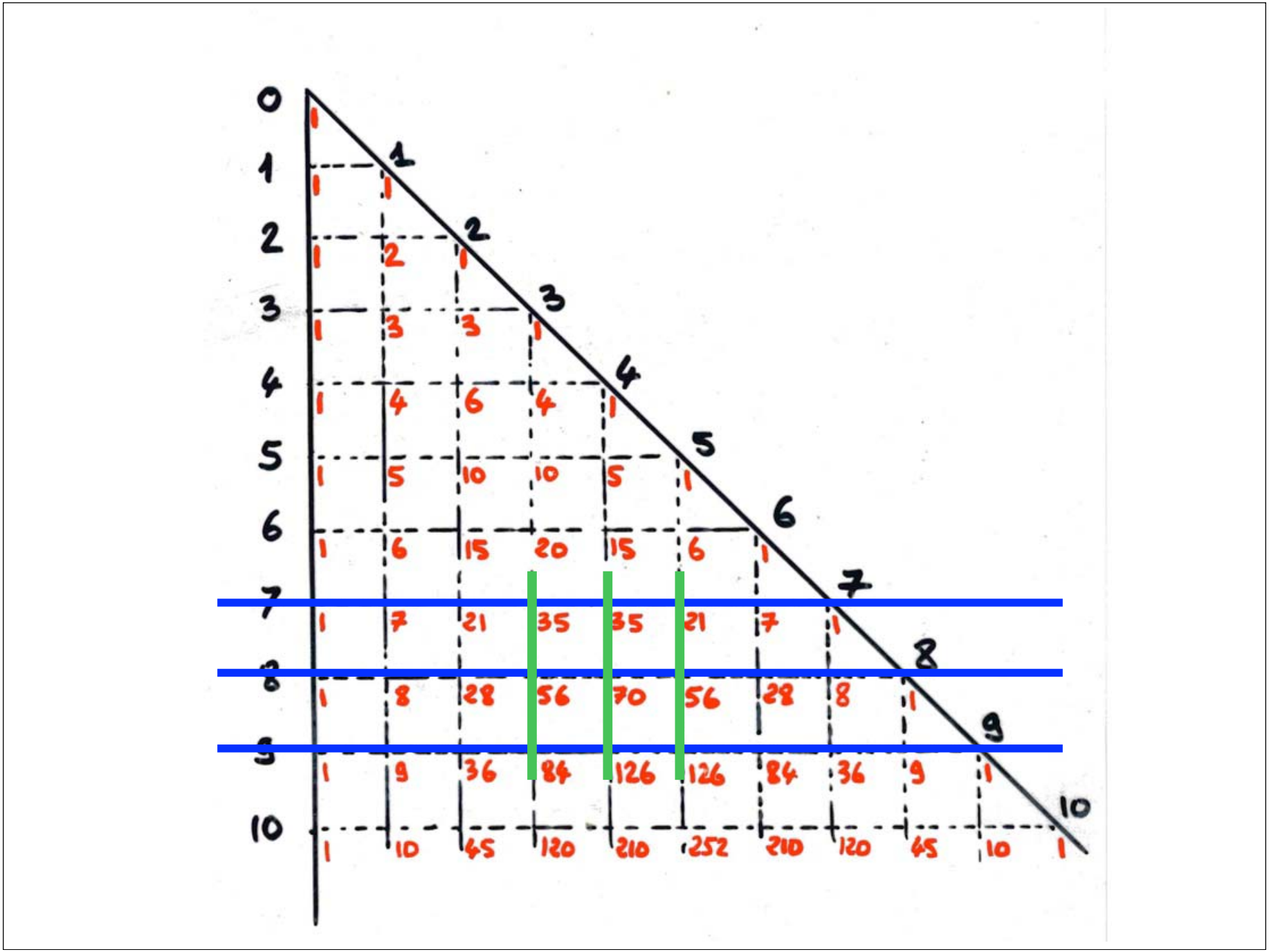
$$\{B_n\}_{n\geq 1} = 1, 2, 22, 96, \ldots$$

© Ministère de l'Éducation nationale

Chung, Graham, Hoggatt, Kleiman (1978)

$$\frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$$

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

Chung, Graham, Hoggatt, Kleiman (1978)

$$B(n) = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$$

© Ministère de l'Éducation nationale

$$B(n+m+1) = \sum_{i,j \geq 0} C_m(i,j)\, T_n(i,j)$$

$$B(n+1) = C_n(0,0)$$

$$C_{n+1}(i,j) = \sum_{0 \leq i' \leq i} C_n(i', j+1) + \sum_{0 \leq j' \leq j} C_n(i+1, j') \qquad n \geq 0$$

$$C_0(i,j) = 1 \qquad i,j \geq 0$$

$$f_{n+1}(x,y) = \sum_{i,j \geq 0} C_n(i,j)\, x^i y^j \qquad , n \geq 0$$

$$f_{n+1}(x,y) = \frac{f_n(x,y) - f_n(x,0)}{(1-x)\,y} + \frac{f_n(x,y) - f_n(0,y)}{(1-y)\,x}$$

$$f_n(x,y) = \sum_{k=1}^{n} \frac{P_{n,k}(x,y)}{(1-x)^{n+1-k}\,(1-y)^k}$$

© Ministère de l'Éducation nationale

$$\begin{cases} P_{n+1,k}(x,y) = \frac{1}{y}\left(P_{n,k}(x,y) - P_{n,k}(x,0)(1-y)^k\right) \\ \qquad + \frac{1}{x}\left(P_{n,k-1}(x,y) - P_{n,k-1}(0,y)(1-x)^{n+2-k}\right) \end{cases}$$

$1 \le n, \quad 1 \le k \le n+1, \quad P_{11}(x,y) = 1, \quad P_{r,s}(x,y) = 0 \quad \begin{matrix} s<0 \\ s>r \end{matrix}$

$$B(n) = C_{n-1}(0,0) = f_n(0,0) = \sum_{k=1}^{n} P_{n,k}(0,0)$$

- $Q_{n,k}(x,y) = P_{n,k}(-x,-y)$
- $Q_{n,k}(x,y) = \sum_{i,j \ge 0} D_{n,k,i,j}\, x^i y^j$

© Ministère de l'Éducation nationale

$$D_{n+1,k,i,j} = \binom{k}{j+1} D_{n,k,i,0} - D_{n,k,i,j+1}$$
$$+ D_{n,k-1,0,j} \binom{n+2-k}{i+1} - D_{n,k-1,i+1,j}$$

$$D_{n,k,i,j} = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \binom{n+1}{k}\binom{n+1}{k+i+1}\binom{n+1}{k-j-1}$$
$$\times \left[\binom{k+i-2}{i}\binom{n+j-k-1}{j} - \binom{k+i-2}{i-1}\binom{n+j-k-1}{j-1} \right]$$

© Ministère de l'Éducation nationale

$$(k-1)k(i+1)(n-k-i)(n-k+j+2)(n-k+j+1)(n-k+j)$$
$$+ (j+1)(k-j-1)(n-k+1)(n-k)(k+i+1)(k+i)(k+i-1)$$
$$+ (n-k+1)(n-k)(i+1)(n-k-i)(k-j-1)\left[i(j+1)-(k-1)(n-k-1)\right]$$
$$+ k(k-1)(j+1)(n-k-i)(k-j-1)\left[j(j+1)-(k-2)(n-k)\right]$$
$$+ (n-1)n(n+1)(i+1)(j+1)\left[ij-(k-1)(n-k)\right] = 0 \quad ?$$

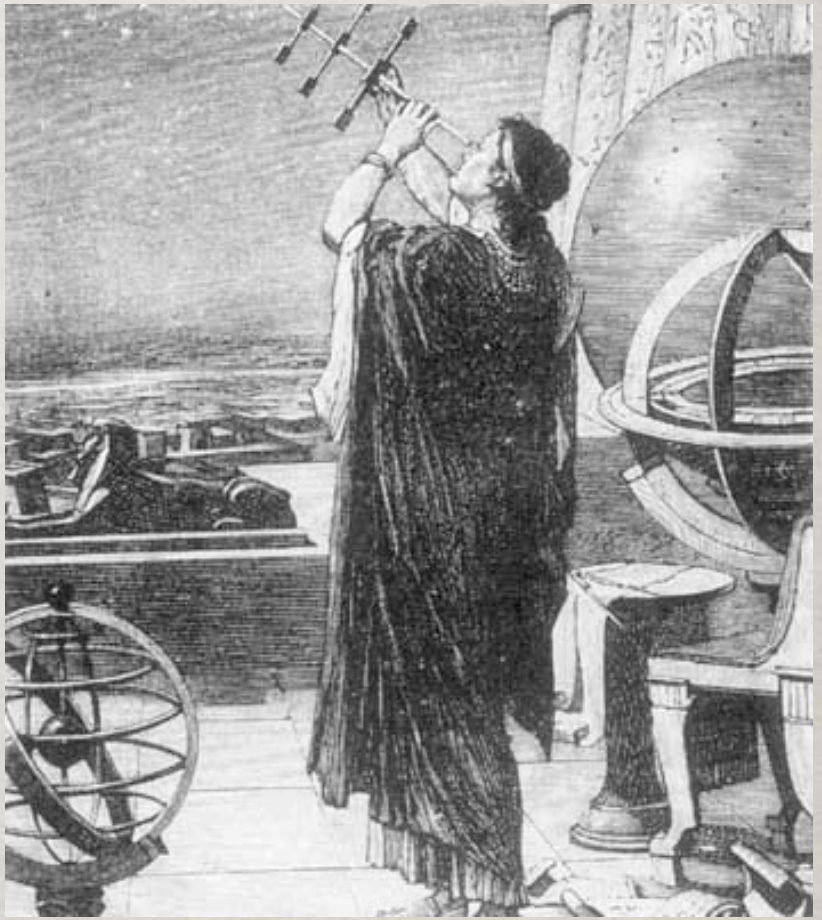
$$D_{n,k,0,0} = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$$

$$B(n) = \sum_{k=1}^{n} D_{n,k,0,0}$$

© Ministère de l'Éducation nationale

© Ministère de l'Éducation nationale

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

M2819 1, 3, 9, 35, 178
Van der Waerden numbers. Ref Loth83 49. [1,2; A5346]

M2820 1, 3, 9, 35, 201, 1827
Coefficients of Bell's formula. Ref NMT 10 65 62. [2,2; A2575, N1134]

M2821 1, 3, 9, 37, 153, 951, 5473, 42729, 353937, 3455083, 30071001, 426685293, 4707929449, 59350096287, 882391484913, 15177204356401, 205119866263713
Sums of logarithmic numbers. Ref TMS 31 79 63. jos. [0,2; A2751, N1135]

M2822 1, 1, 1, 3, 9, 37, 177, 959, 6097, 41641, 325249, 2693691, 24807321, 241586893, 2558036145, 28607094455, 342232522657, 4315903789009, 57569080467073
Expansion of $e^{\tan x}$. Ref JO61 150. [0,4; A6229]

M2823 1, 3, 9, 42, 206, 1352, 10168
Regular semigroups of order n. Ref PL65. MAL 2 2 67. SGF 14 71 77. [1,2; A1427, N1136]

M2824 0, 1, 1, 3, 9, 45, 225, 1575, 11025, 99225, 893025, 9823275, 108056025, 1404728325, 18261468225, 273922023375, 4108830350625, 69850115960625
Expansion of $1 / (1-x)(1-x^2)^{1/2}$. Ref R1 87. [1,4; A0246, N1137]

M2825 1, 1, 1, 3, 9, 48, 504, 14188, 1351563
Threshold functions of n variables. Ref PGEC 19 821 70. MU71 38. [0,4; A1530, N1138]

M2826 3, 9, 54, 450, 4725, 59535, 873180, 14594580
Expansion of an integral. Ref C1 167. [2,1; A1194, N1139]

M2827 1, 3, 9, 89, 1705, 67774
Superpositions of cycles. Ref AMA 131 143 73. [3,2; A3225]

M2828 1, 3, 9, 93, 315, 3855, 13797, 182361, 9256395, 34636833, 1857283155, 26817356775, 102280151421, 1497207322929, 84973577874915, 4885260612740877
Fermat quotients: $(2^{p-1}-1)/p$. Ref Well86 70. [0,2; A7663]

M2829 3, 10, 4, 5, 10, 2, 5, 3, 2, 3, 6, 6, 6, 3, 5, 6, 10, 5, 5, 10, 6, 6, 6, 2, 5, 8, 2, 6, 8, 4, 6, 6, 4, 5, 10, 2, 4, 7, 11, 5, 7, 9, 10, 7, 1, 6, 7, 11, 7, 10, 0, 6, 8, 9, 6, 4, 11, 7, 13, 2, 6, 4, 4
Iterations until $3n$ reaches 153 under x goes to sum of cubes of digits map. Ref Robe92 13. [1,1; A3620]

M2830 1, 3, 10, 13, 63, 75, 137, 543, 949, 6206, 13361, 73011, 597449, 1865358,

© Ministère de l'Éducation nationale

21489003, 80198051, 299303201, 1117014753, 4168755811, 15558008491
$a(n)=4a(n-1)-a(n-2)$. Ref EUL (1) 1 375 11. MMAG 40 78 67. [0,3; A1835, N1160]

M2895 1, 3, 11, 43, 171, 683, 2731, 10923, 43691, 174763, 699051, 2796203, 11184811, 44739243, 178956971, 715827883, 2863311531, 11453246123, 45812984491
$(2^{2n+1}+1)/3$. Ref JGT 17 625 93. [0,2; A7583]

M2896 3, 11, 43, 683, 2731, 43691, 174763, 2796203, 715827883, 2932031007403, 768614336404564651, 201487636602438195784363
Primes of form $(2^p+1)/3$. Ref MMAG 27 157 54. [1,1; A0979, N1161]

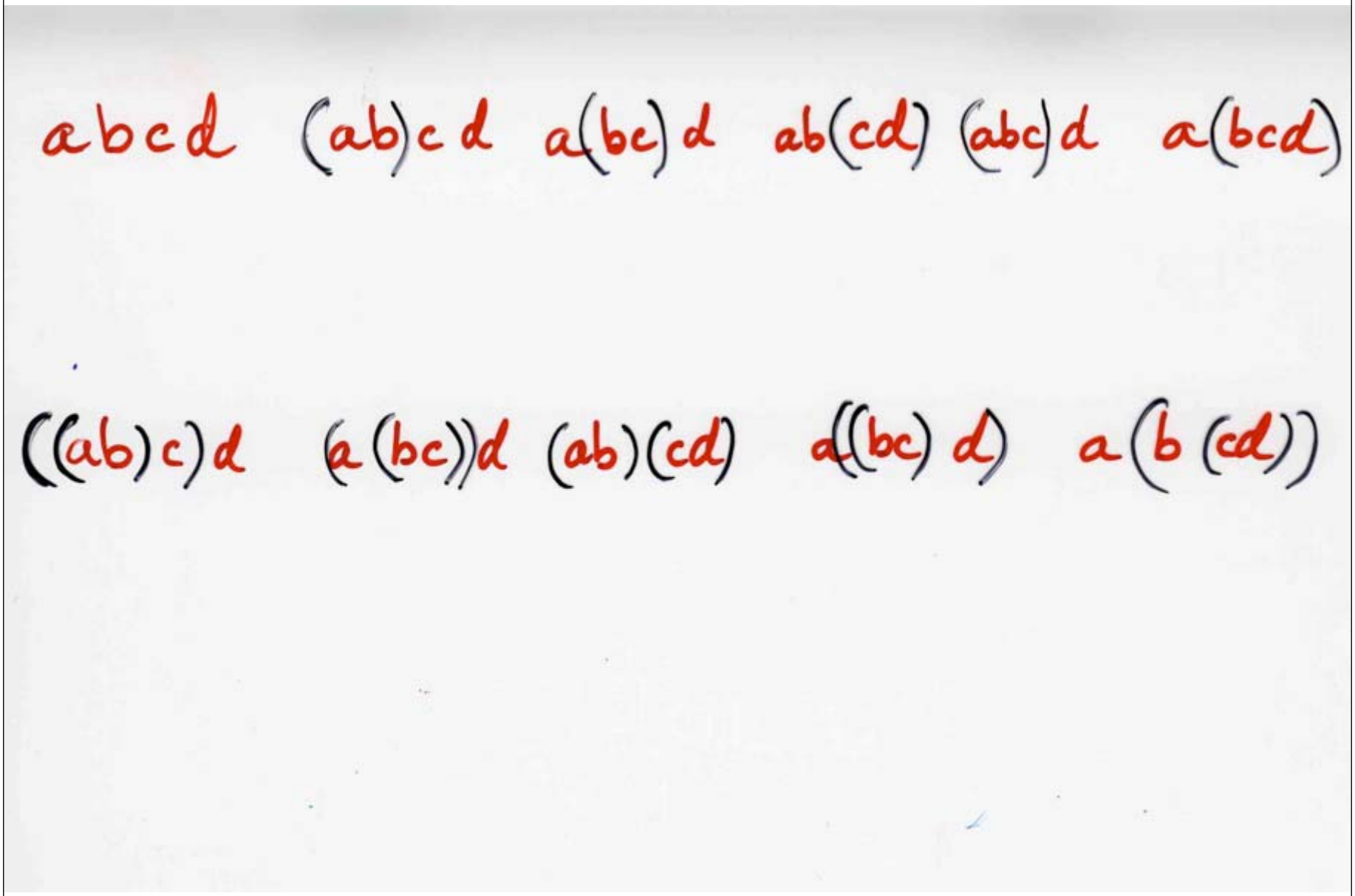
M2897 1, 3, 11, 44, 186, 814, 3652, 16689, 77359, 362671, 1716033, 8182213, 39267086, 189492795, 918837374, 4474080844, 21866153748, 107217298977, 527266673134
Fixed hexagonal polyominoes with n cells. Ref RE72 97. dhr. [1,2; A1207, N1162]

M2898 1, 1, 3, 11, 45, 197, 903, 4279, 207[illegible], 103049, [illegible]18859, 2646723, 13648869, 71039373, 372693519, 1968801519, 1046[illegible]5909013009, 300159426963
Schroeder's second problem: $(n+1)a(n+[illegible]-1)a(n)-(n-2)a(n-1)$. Ref EDMN 32 6 40. BAMS 54 359 48. RCI 168. C[illegible]198. [1,3; A1003, N1163]

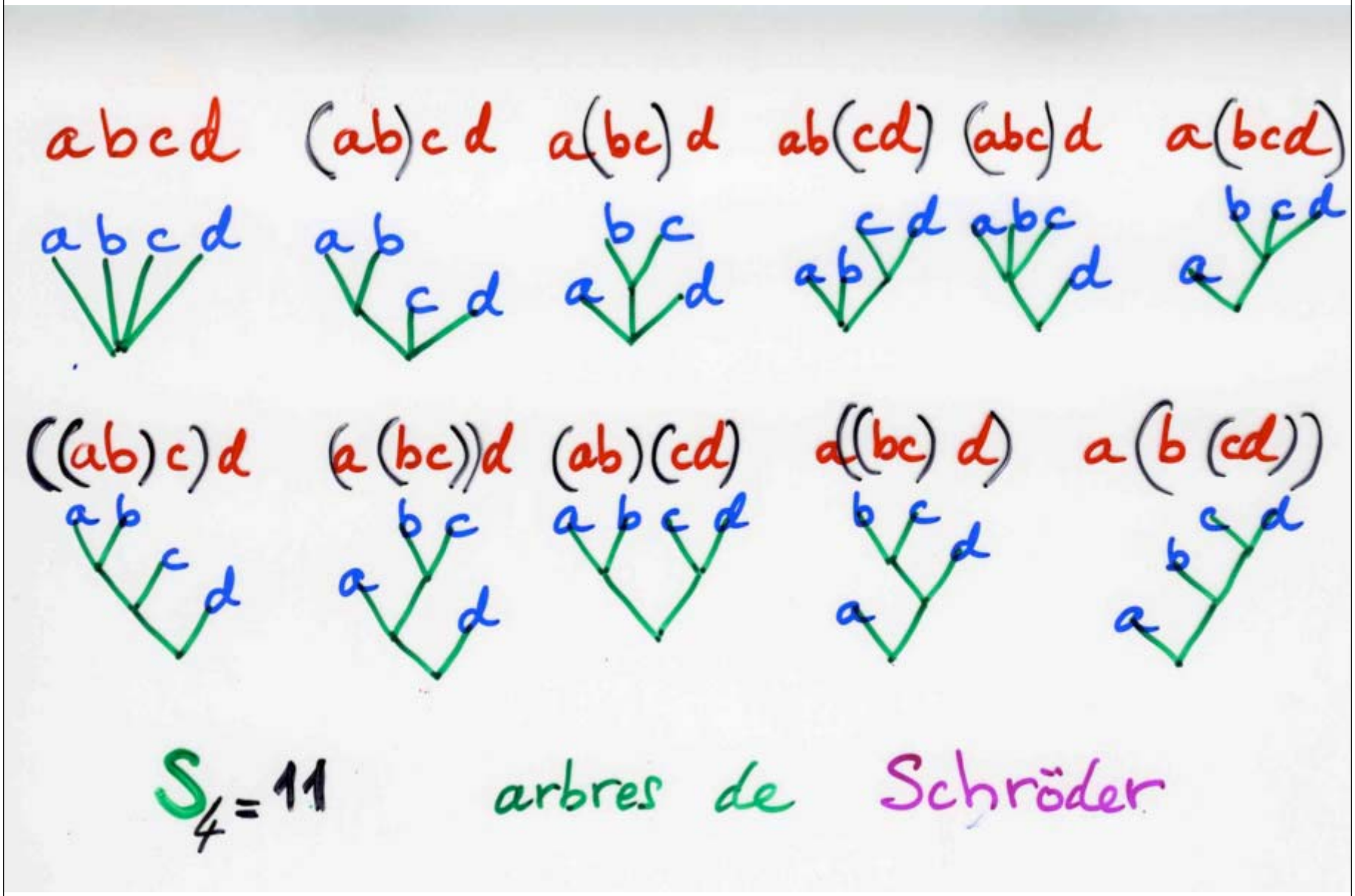
© Ministère de l'Éducation nationale

Schröder

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

Chaînes de spins quantiques ....

© Ministère de l'Éducation nationale

Spin chains and combinatorics

A. V. Razumov, Yu. G. Stroganov

Institute for High Energy Physics
142284 Protvino, Moscow region, Russia

The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

© Ministère de l'Éducation nationale

The nonzero wave function components are

$$N = 3: \quad \psi_{001} = 1;$$
$$N = 5: \quad \psi_{00011} = 1, \quad \psi_{00101} = 2;$$
$$N = 7: \quad \psi_{0000111} = 1, \quad \psi_{0001101} = \psi_{0001011} = 3, \quad \psi_{0010011} = 4 \quad \psi_{0010101} = 7.$$

All components not included in the list can be obtained by shifting. Notice that the components of the ground state are positive in accordance with the Perron–Frobenius theorem.

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$\psi_{000001111} = 1, \quad \psi_{000010111} = 4, \quad \psi_{000011011} = 6, \quad \psi_{000100111} = 7,$$
$$\psi_{000101011} = 17, \quad \psi_{000101101} = 14, \quad \psi_{000110011} = 12, \quad \psi_{001001011} = 21,$$
$$\psi_{001010011} = 25, \quad \psi_{001010101} = 42.$$

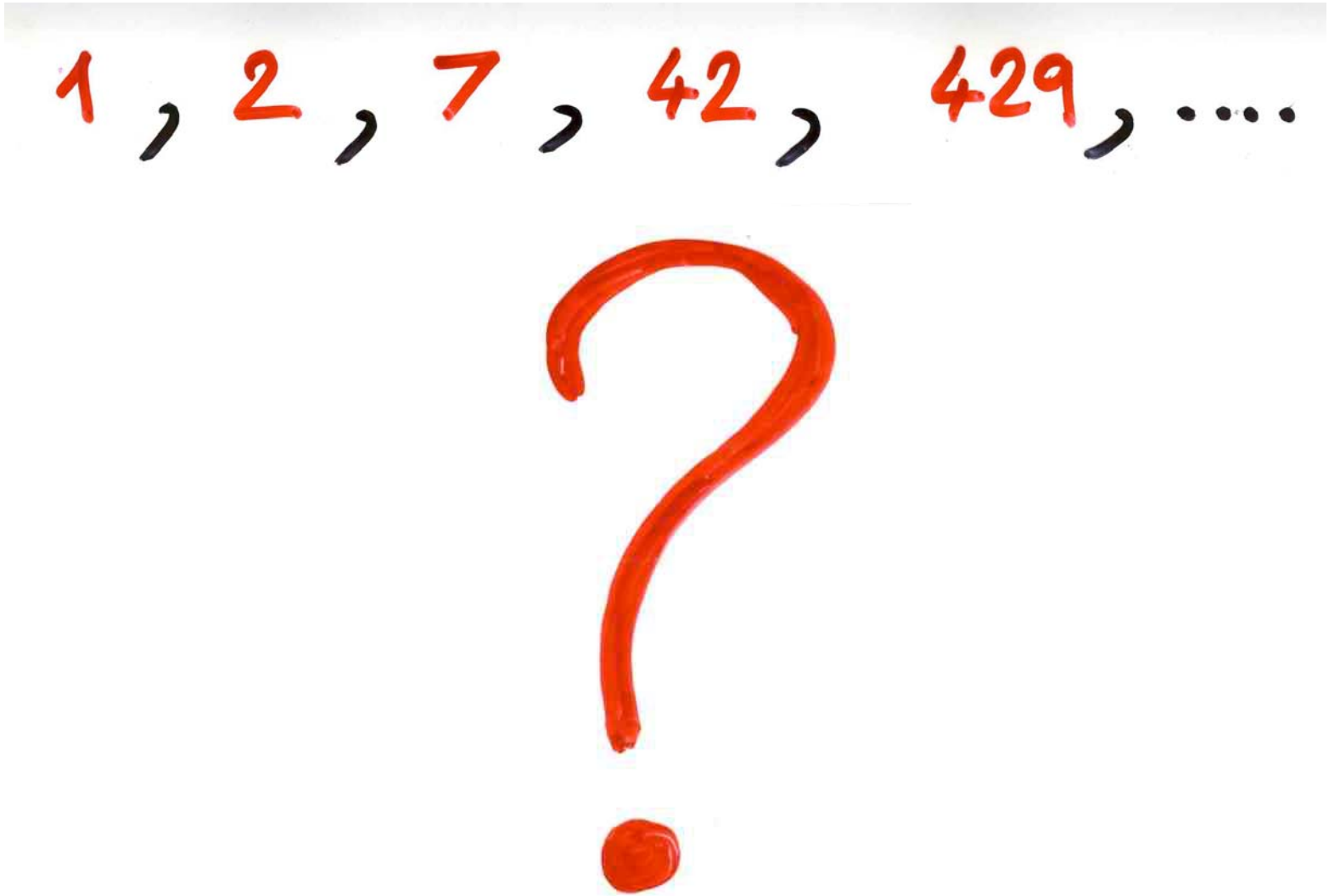
Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$\psi_{000001111} = 1, \quad \psi_{000010111} = 4, \quad \psi_{000011011} = 6, \quad \psi_{000100111} = 7,$$
$$\psi_{000101011} = 17, \quad \psi_{000101101} = 14, \quad \psi_{000110011} = 12, \quad \psi_{001001011} = 21,$$
$$\psi_{001010011} = 25, \quad \psi_{001010101} = 42.$$

We omit nonzero components which can be obtained by the reflection of the order of sites since this transformation is a symmetry of our state, as it is for the ground state. For example, we have

$$\psi_{000011101} = \psi_{000010111} = 4.$$

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

M1803 1, 2, 7, 37, 266, 2431, 27007, ...

M1791 0, 1, 2, 7, 32, 181, 1214, 9403, 82508, 808393, 8743994, 103459471, 1328953592, 18414450877, 273749755382, 4345634192131, 73362643649444, 1312349454922513
$a(n)=n.a(n-1)+(n-2)a(n-2)$. Ref R1 188. [0,3; A0153, N0706]

E.g.f.: $(1-x)^{-3}\,e^{-x}$.

M1792 1, 1, 2, 7, 32, 181, 1232, 9787, 88832, 907081, 10291712, 128445967, 1748805632, 25794366781, 409725396992, 6973071372547, 126585529106432
Expansion of $1/(1-\sinh x)$. Ref ARS 10 138 80. [0,3; A6154]

M1793 0, 1, 1, 2, 7, 32, 184, 1268, 10186, 93356, 960646, 10959452, 137221954, 1870087808, 27548231008, 436081302248, 7380628161076, 132975267434552
Stochastic matrices of integers. Ref DUMJ 35 659 68. [0,4; A0987, N0707]

M1794 1, 2, 7, 33, 192
Permutations of length n with n in second orbit. Ref C1 258. [2,2; A6595]

M1795 1, 2, 7, 34, 209, 1546, 13327, 130922, 1441729, 17572114, 234662231, 3405357682, 53334454417, 896324308634, 16083557845279, 306827170866106
$a(n)=2n.a(n-1)-(n-1)^2a(n-2)$. Ref SE33 78. [0,2; A2720, N0708]

M1796 1, 2, 7, 34, 257, 2606, 32300, 440564, 6384634
Polyhedra with n nodes. Ref GR67 424. UPG B15. Dil92. [4,2; A0944, N0709]

M1797 2, 7, 35, 219, 1594, 12935, 113945, 1070324, 10586856, 109259633, 1168384157, 12877168147, 145656436074, 1685157199175, 19886174611045
Two-rowed truncated monotone triangles. Ref JCT A42 277 86. Zei93. [1,1; A6947]

M1798 1, 1, 2, 7, 35, 228, 1834, 17382, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Coefficients of iterated exponentials. Ref SMA 11 353 45. [0,3; A0154, N0710]

M1799 1, 2, 7, 35, 228, 1834, 17582, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Expansion of $\ln(1+\ln(1+x))$. [0,2; A3713]

M1800 1, 0, 1, 2, 7, 36, 300, 3218, 42335, 644808
Circular diagrams with n chords. Ref BarN94. [0,4; A7474]

M1801 1, 2, 7, 36, 317, 5624, 251610, 33642660, 14685630688
$n \times n$ binary matrices. Ref CPM 89 217 64. SLC 19 79 88. [0,2; A2724, N0711]

M1802 2, 7, 37, 216, 1780, 32652
Semigroups of order n with 2 idempotents. Ref MAL 2 2 67. SGF 14 71 77. [2,1; A2787, N0712]

M1803 1, 2, 7, 37, 266, 2431, 27007, 353522, 5329837, 90960751, 1733584106, 36496226977, 841146804577, 21065166341402, 569600638022431
$a(n)=(2n-1)a(n-1)+a(n-2)$. Ref RCI 77. [0,2; A1515, N0713]

© Ministère de l'Éducation nationale

M1804 1, 1, 2, 7, 38, 291, 2932, 36961, 561948, 10026505, 205608536, 4767440679, 123373203208, 3525630110107, 110284283006640, 3748357699560961
Forests of labeled trees with n nodes. Ref JCT 5 96 68. SIAD 3 574 90. [0,3; A1858, N0714]

M1805 1, 1, 2, 7, 40, 357, 4824, 96428, 2800472, 116473461
n-element partial orders contained in linear order. Ref nbh. [0,3; A6455]

M1806 1, 2, 7, 41, 346, 3797, 51157, 816356, 15050581, 314726117, 7359554632, 190283748371, 5389914888541, 165983936096162, 5521346346543307
Planted binary phylogenetic trees with n labels. Ref LNM 884 196 81. [1,2; A6677]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \ldots n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1810 0, 1, 2, 7, 44, 361, 3654, 44207, 622552, 10005041, 180713290, 3624270839, 79914671748, 1921576392793, 50040900884366, 1403066801155039
Modified Bessel function $K_n(1)$. Ref AS1 429. [0,3; A0155, N0716]

M1811 0, 1, 2, 7, 44, 447, 6749, 142176, 3987677, 143698548, 6470422337, 356016927083, 23503587609815, 1833635850492653, 166884365982441238
$a(n)=n(n-1)a(n-1)/2+a(n-2)$. [0,3; A1046, N0717]

M1812 1, 2, 7, 44, 529, 12278, 565723, 51409856, 9371059621, 3387887032202, 2463333456292207, 3557380311703796564, 1033908166635018028984 9
Sum of Gaussian binomial coefficients $[n,k]$ for $q=4$. Ref TU69 76. GJ83 99. ARS A17 328 84. [0,2; A6118]

M1813 2, 7, 52, 2133, 2590407, 3374951541062, 5695183504479116640376509, 162175575749223863014205141915237848956395777104 80
Free binary trees of height n. Ref JCIS 17 180 92. [1,1; A5588]

M1814 1, 1, 2, 7, 56, 2212, 2595782, 3374959180831, 5695183504489239067484387, 162175575749223863014205312770713651031687342842 82
Planted 3-trees of height n. Ref RSE 59(2) 159 39. CMB 11 87 68. JCIS 17 180 92. [0,3; A2658, N0718]

© Ministère de l'Éducation nationale

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \ldots n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

© Ministère de l'Éducation nationale

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \ldots n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

© Ministère de l'Éducation nationale

MSA

Matrices à signes alternants

© Ministère de l'Éducation nationale

"Matrices à signes alternés"

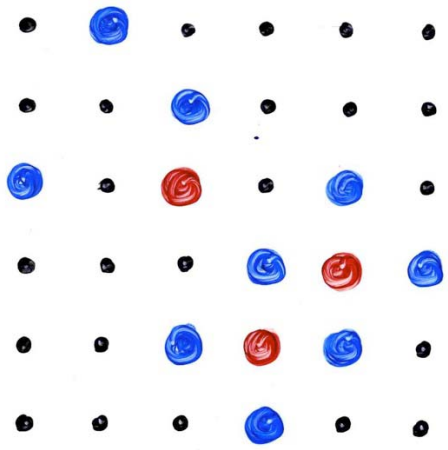
(alternating sign matrices)

- entrées : 0, 1, −1
- somme des entrées, ligne, colonne = 1
- entrées ≠ 0 alternent en signe ligne, colonne

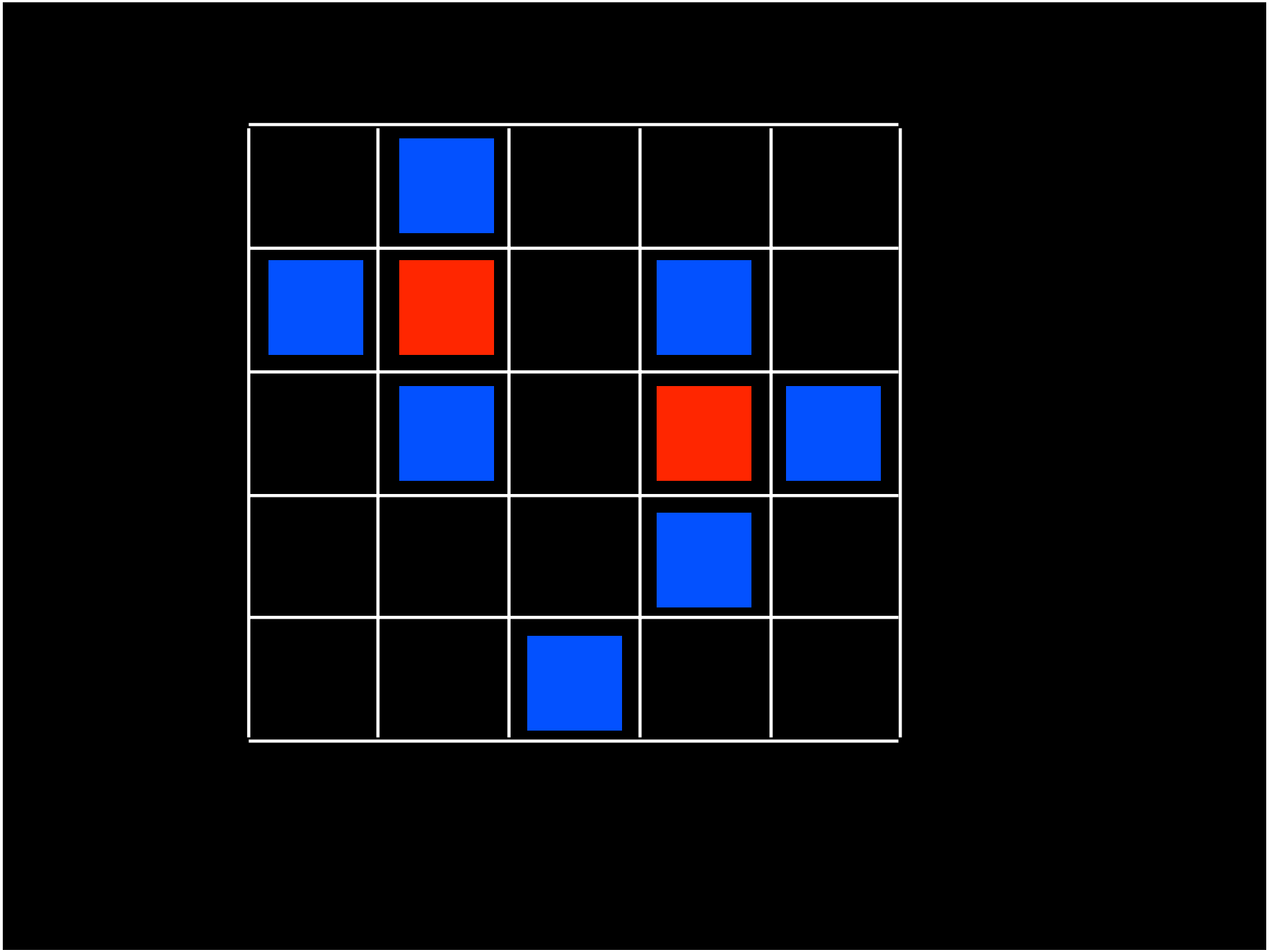
alternating sign matrices (ASM)

ex :

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 6 \text{ permutations}$$

1, 2, 7, 42, 429, ...

© Ministère de l'Éducation nationale

MARS

"Physique combinatoire:
autour des matrices à signes alternants
et la conjecture de Razumov - Stroganov"

Projet Blanc ANR
LaBRI , IGM

© Ministère de l'Éducation nationale

Enumérer les MSA

© Ministère de l'Éducation nationale

$$1, 2, 7, 42, 429, \ldots$$

$$\frac{1!\ 4!\ 7!\ \cdots\ (3n-2)!}{n!\ (n+1)!\ (n+2)!\ \cdots\ (n+n-1)!}$$

© Ministère de l'Éducation nationale

deviner une formule

$$a_n = \frac{(\text{produit})}{(\text{produit})}$$

RATE

C. Kratzenthaler

M. Rubey

© Ministère de l'Éducation nationale

$b_n = \dfrac{a_{n+1}}{a_n}$.

rat(n)

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{(n+2)}$$

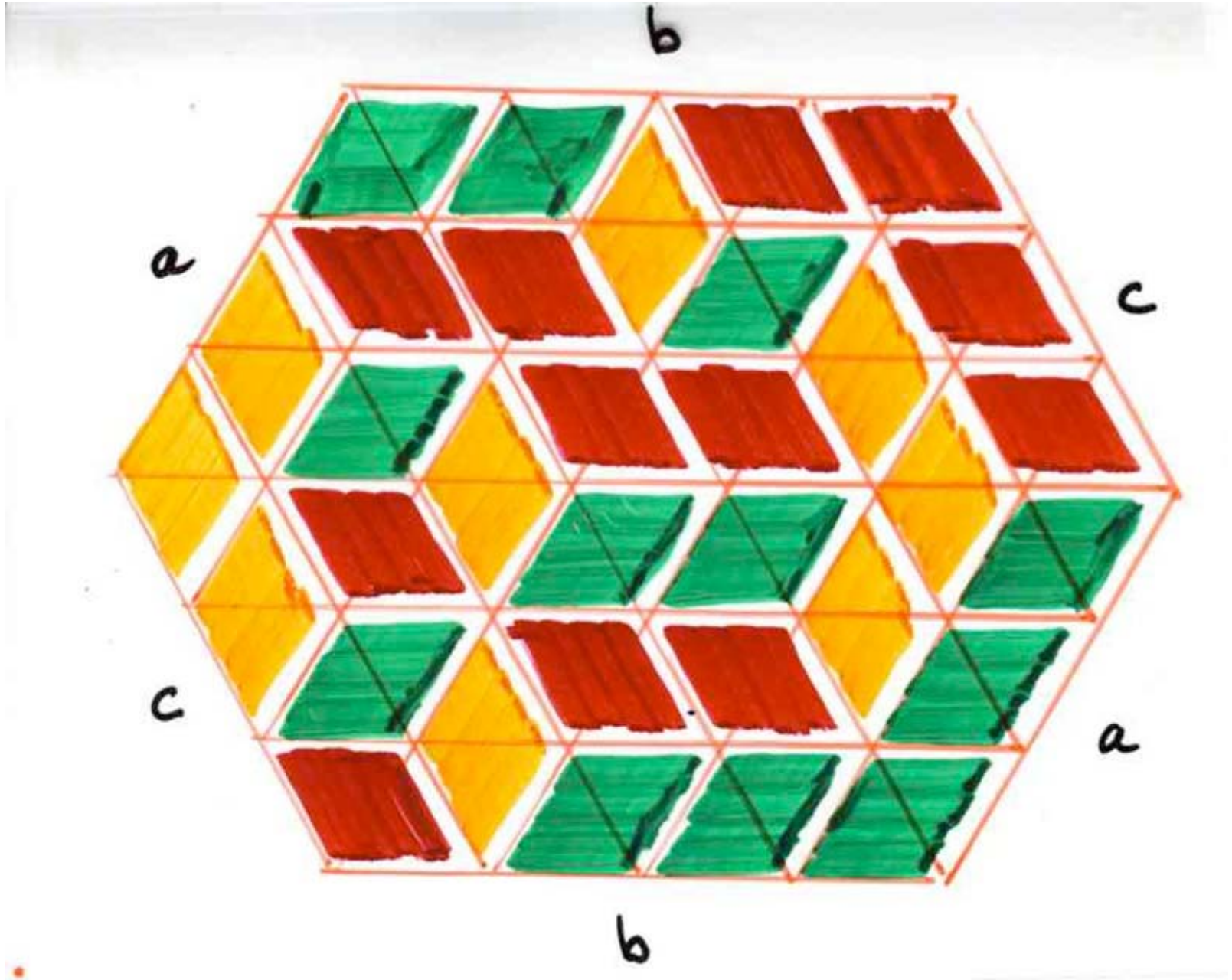
Catalan

$$c_n = \frac{b_{n+1}}{b_n}$$

© Ministère de l'Éducation nationale

Pavages

© Ministère de l'Éducation nationale



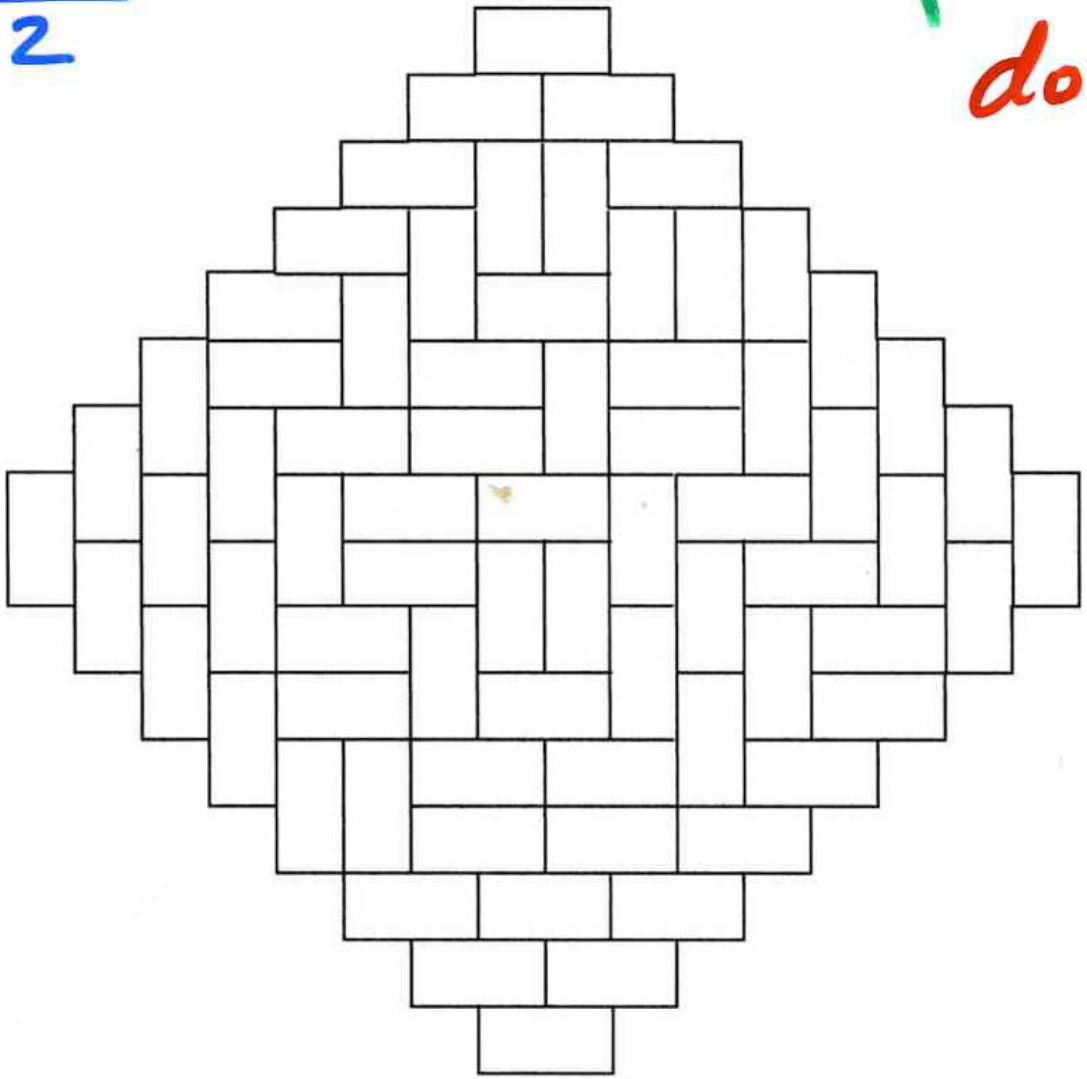
© Ministère de l'Éducation nationale

$$\prod_{\substack{1 \leqslant i \leqslant a \\ 1 \leqslant j \leqslant b \\ 1 \leqslant k \leqslant c}} \frac{i+j+k-1}{i+j+k-2}$$

© Ministère de l'Éducation nationale

$2^{\frac{n(n-1)}{2}}$

pavage dominos

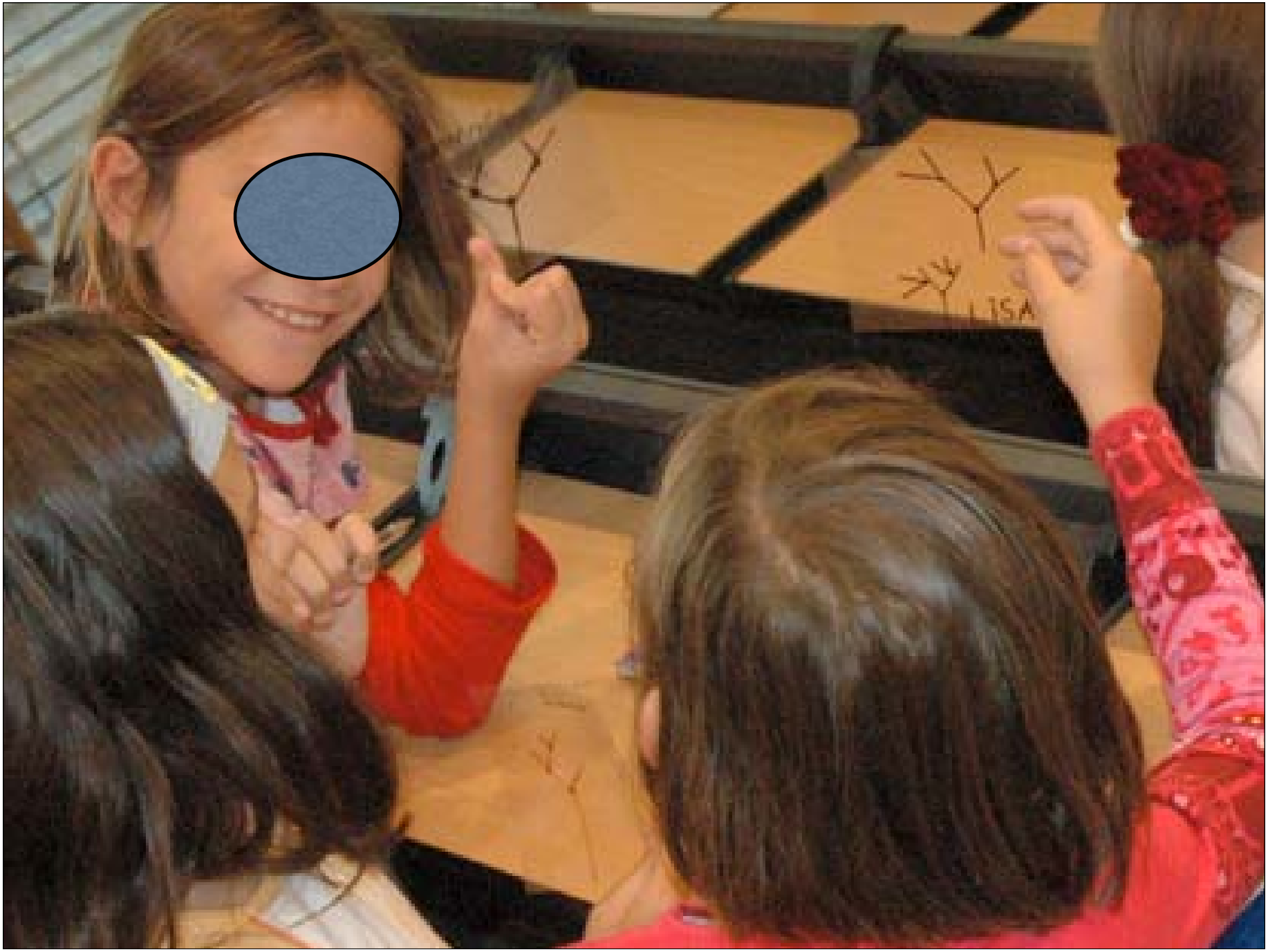


© Ministère de l'Éducation nationale

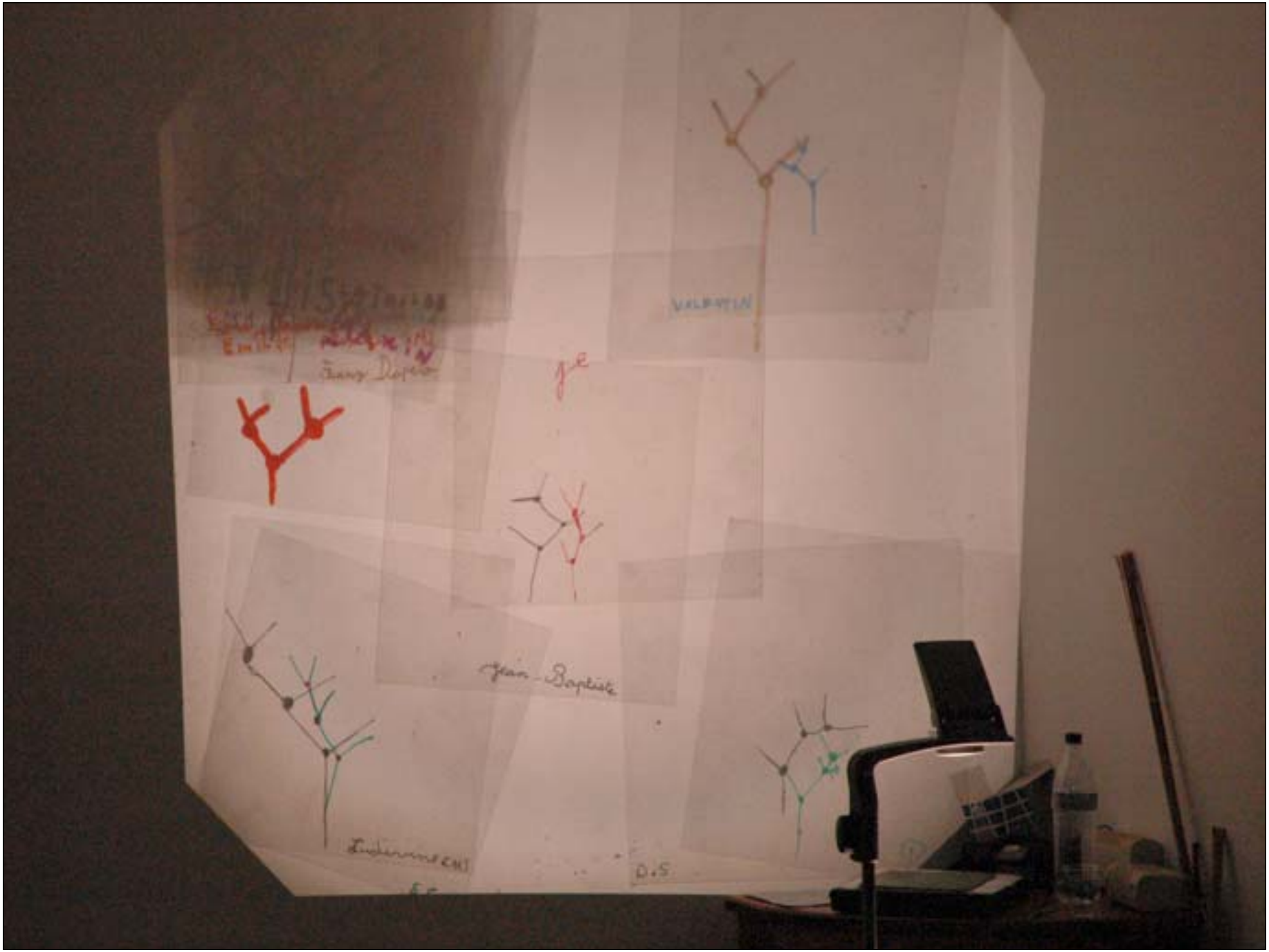
Addition, soustraction, division, ..
les nombres de Catalan
à l'école primaire

© Ministère de l'Éducation nationale

© Ministère de l'Éducation nationale



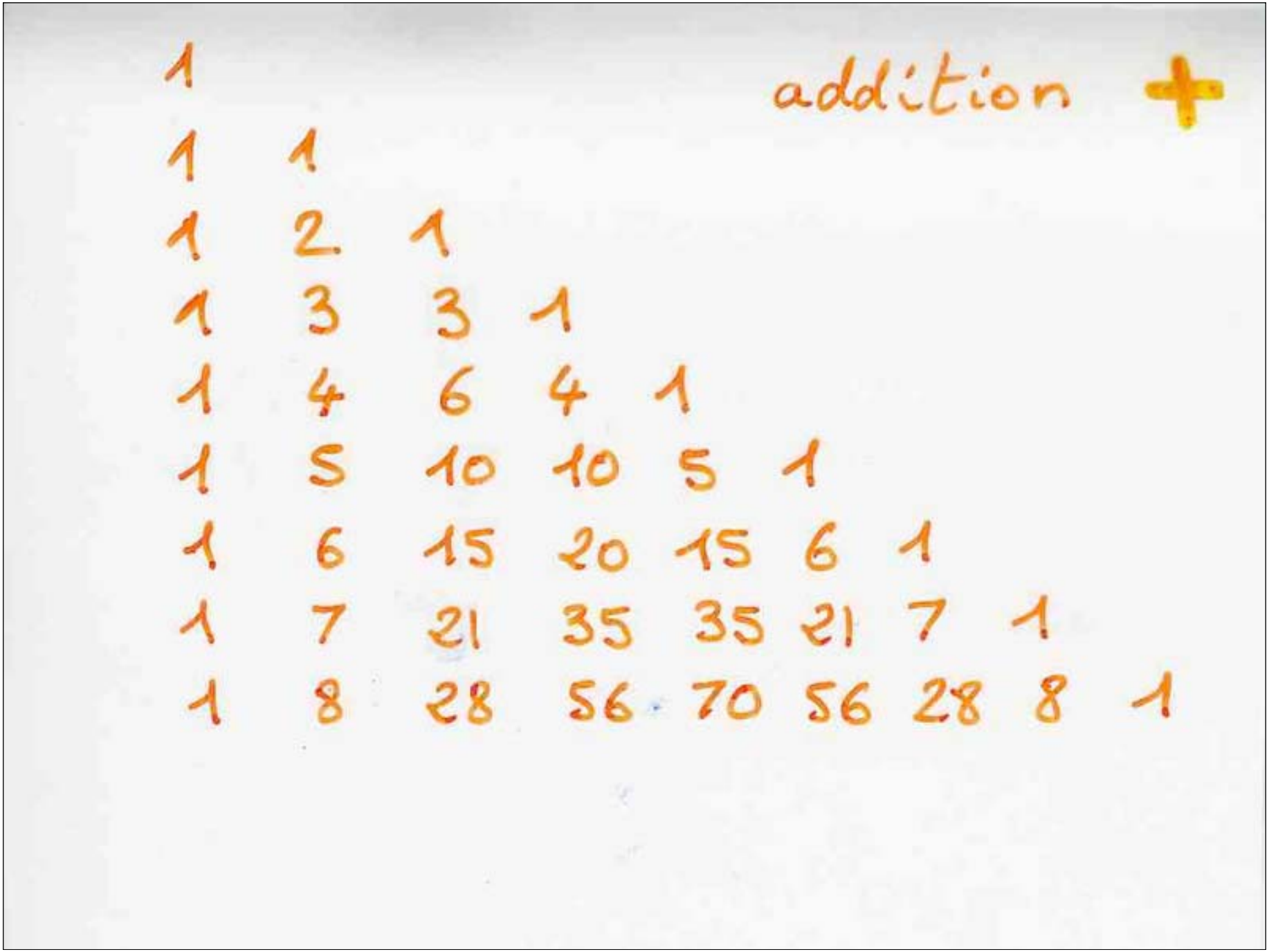
© Ministère de l'Éducation nationale



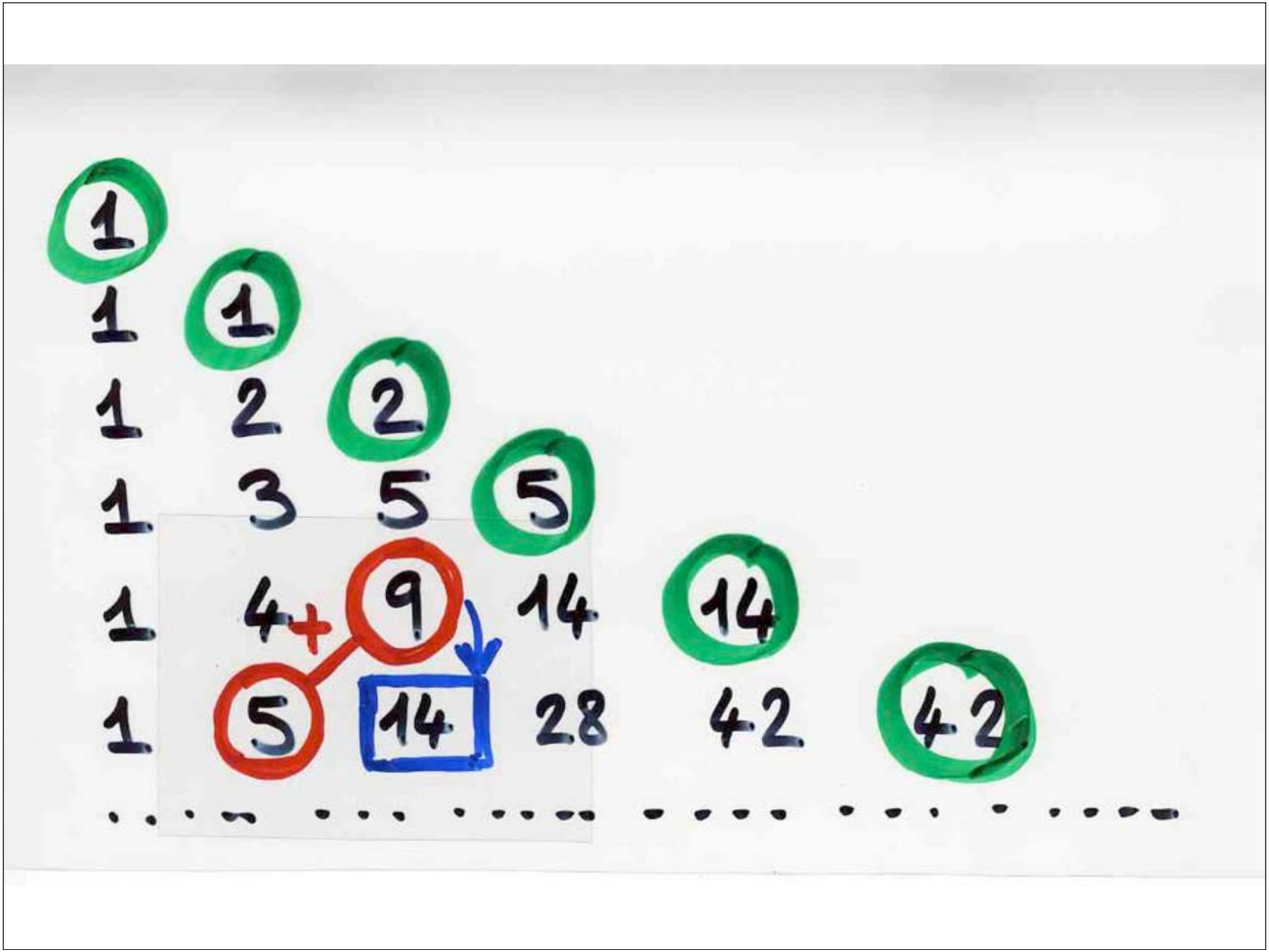
© Ministère de l'Éducation nationale

Addition +

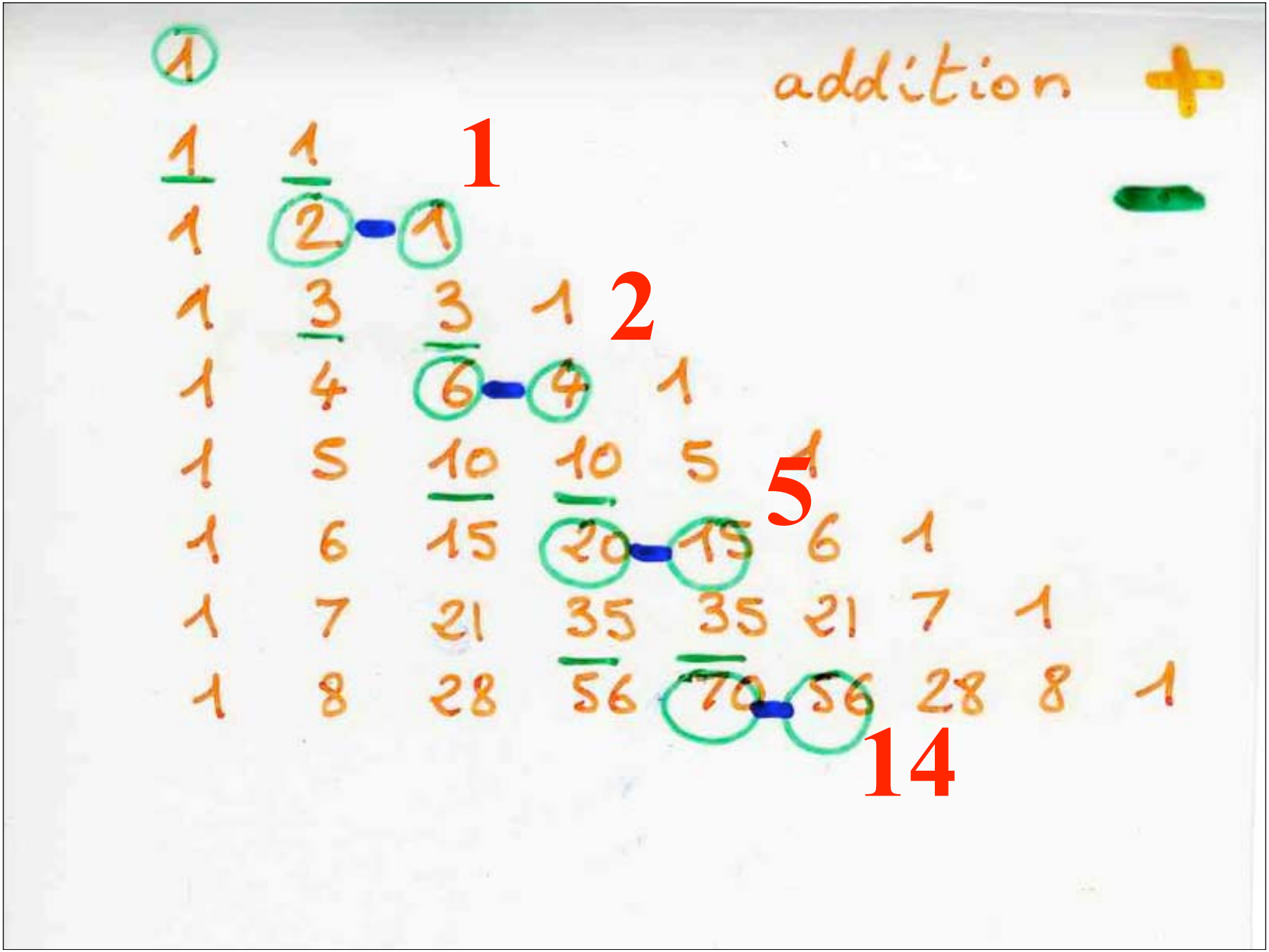
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

© Ministère de l'Éducation nationale

expérimentations avec les séries génératrices

© Ministère de l'Éducation nationale

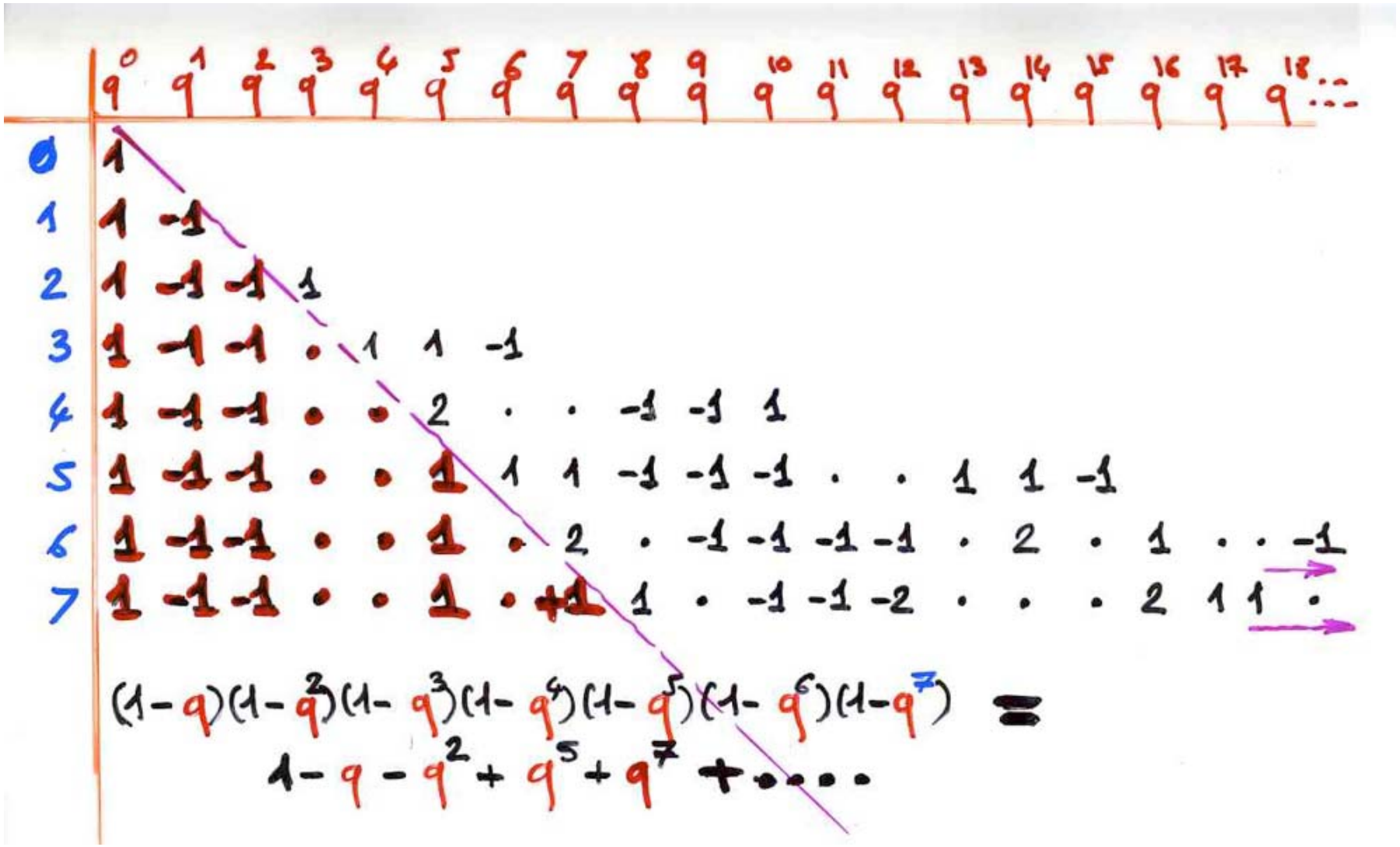
$(1-q) =$

$(1-q)(1-q^2) =$

$(1-q)(1-q^2)(1-q^3) =$

$(1-q)(1-q^2)(1-q^3)(1-q^4) =$

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

EVOLVTIO
PRODVCTI INFINITI
$(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$
IN SERIEM SIMPLICEM.

Auctore
L. EVLERO.

© Ministère de l'Éducation nationale

$$(1-q)\cdots(1-q^{15}) = 1-q-q^2+q^5+q^7-q^{12}-q^{15}+\ldots$$

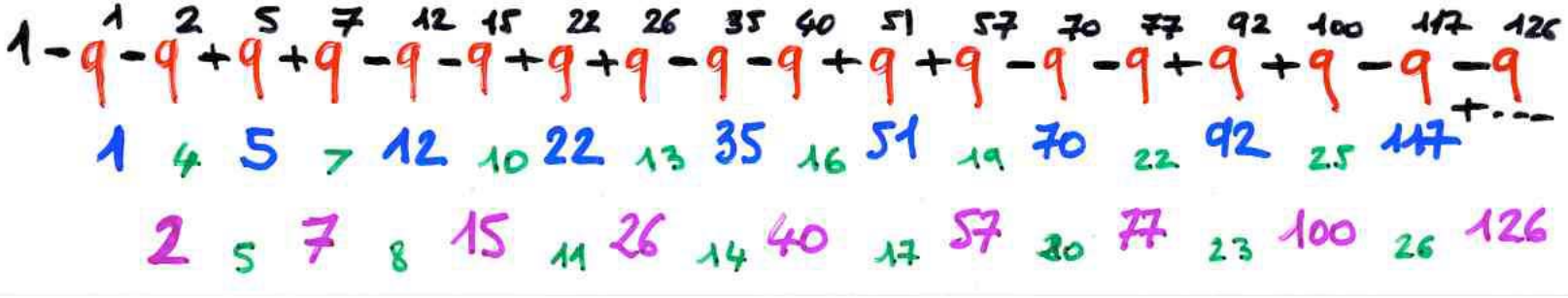
$$1-q^{1}-q^{2}+q^{5}+q^{7}-q^{12}-q^{15}+q^{22}+q^{26}-q^{35}-q^{40}+q^{51}+q^{57}-q^{70}-q^{77}+q^{92}+q^{100}-q^{117}-q^{126}+\ldots$$

$$1-q^{1}-q^{2}+q^{5}+q^{7}-q^{12}-q^{15}+q^{22}+q^{26}-q^{35}-q^{40}+q^{51}+q^{57}-q^{70}-q^{77}+q^{92}+q^{100}-q^{117}-q^{126}+\ldots$$

1 5 12 22 35 51 70 92 117

2 7 15 26 40 57 77 100 126

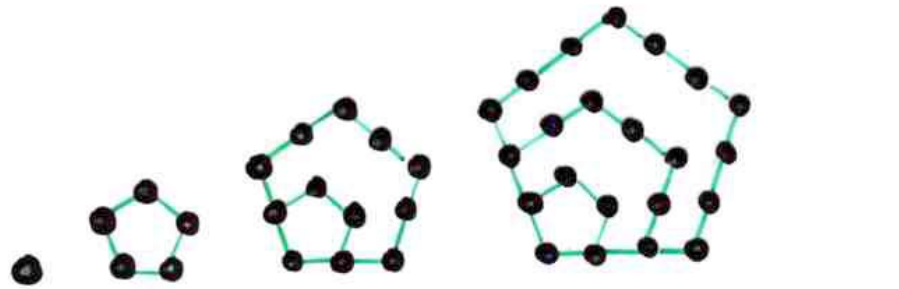
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

$$\prod_{i \geqslant 1} (1 - q^i) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots$$

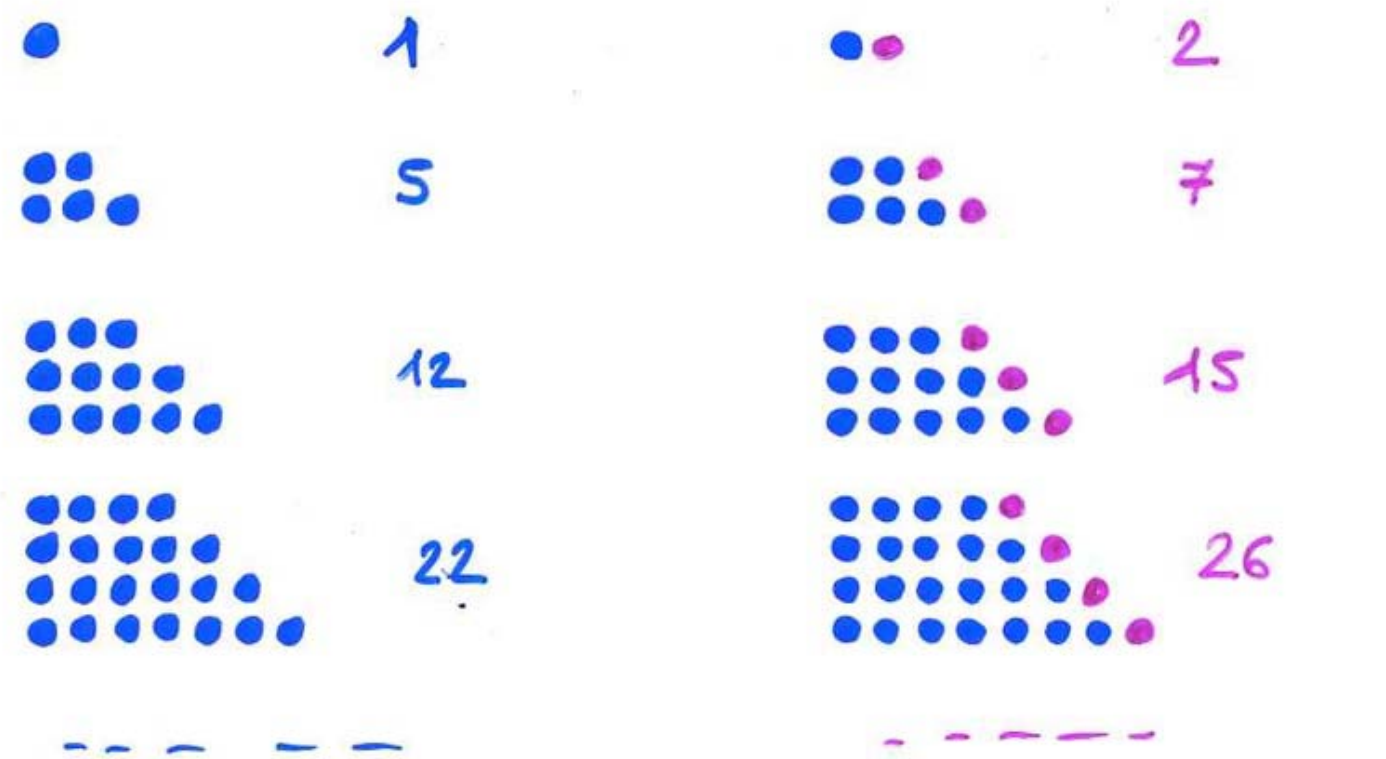
$$= \sum_{n \geqslant 1} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2} \right)$$



1 5 12 22 ...

nombres pentagonaux $n(3n-1)/2$

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

deviner une formule
pour une série génératrice

© Ministère de l'Éducation nationale

séries rationnelles $\sum_{n\geq 0} a_n t^n = \frac{N(t)}{D(t)}$

ni

séries algébriques $P(y,t) = 0$

ni

séries P-récursives

$$P_k(n)\,a_{n+k} + P_{k-1}(n)\,a_{n+k-1} + \cdots + P_0(n)\,a_n = 0$$

© Ministère de l'Éducation nationale

gfun

B. Salvy

P. Zimmermann

(1994)

© Ministère de l'Éducation nationale

Approximants de Padé

$$f(t) = a_0 + a_1 t + \quad + a_n t^n + \ldots$$

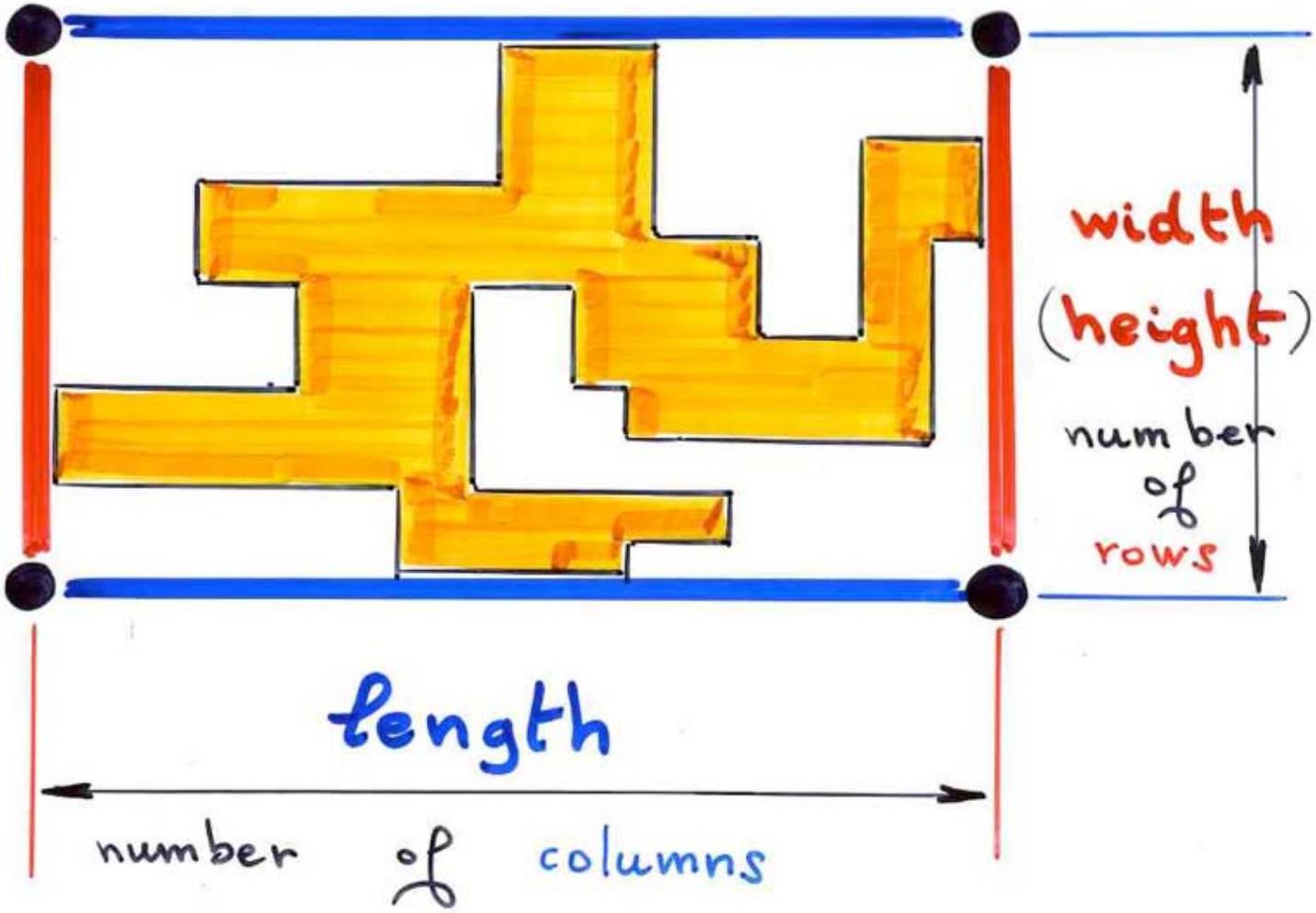
$$\simeq \frac{N_p(t)}{D_q(t)}$$

Padé algébriques

© Ministère de l'Éducation nationale

polyominos convexes

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

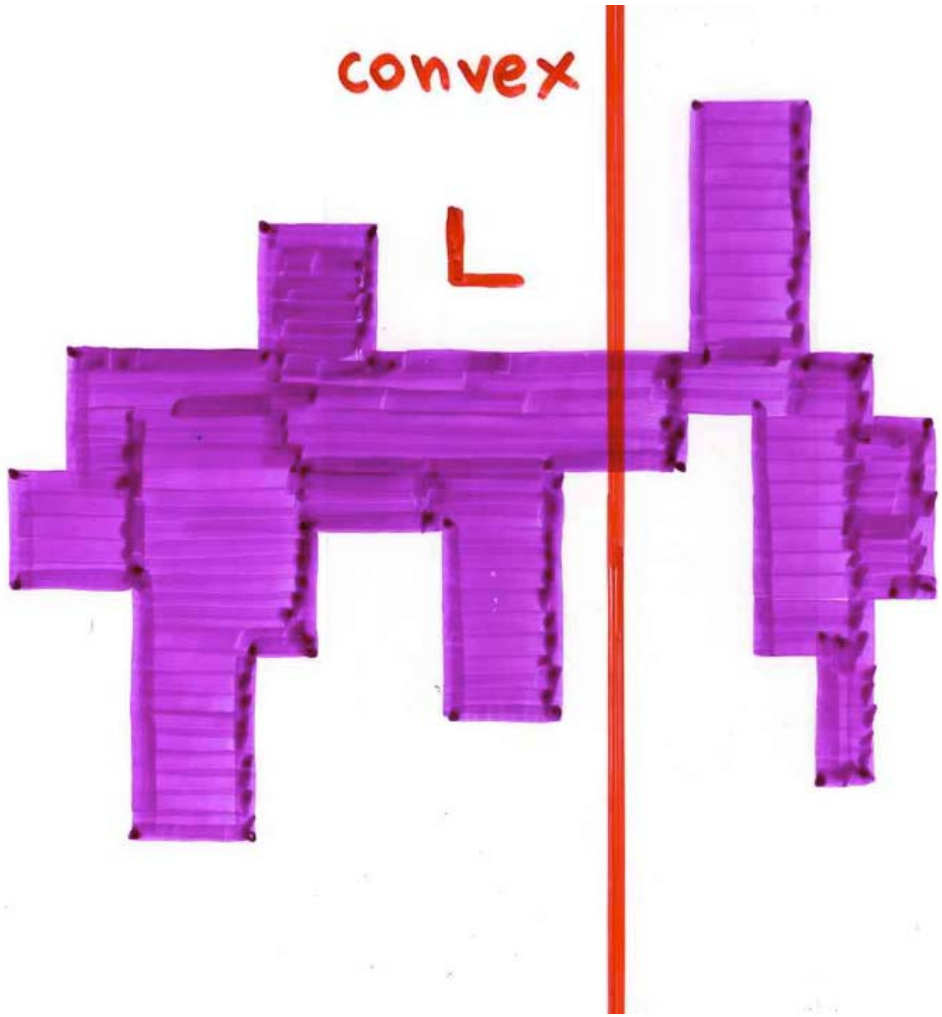
Polyominoes enumeration

a_n = number of polyominoes with **area** n

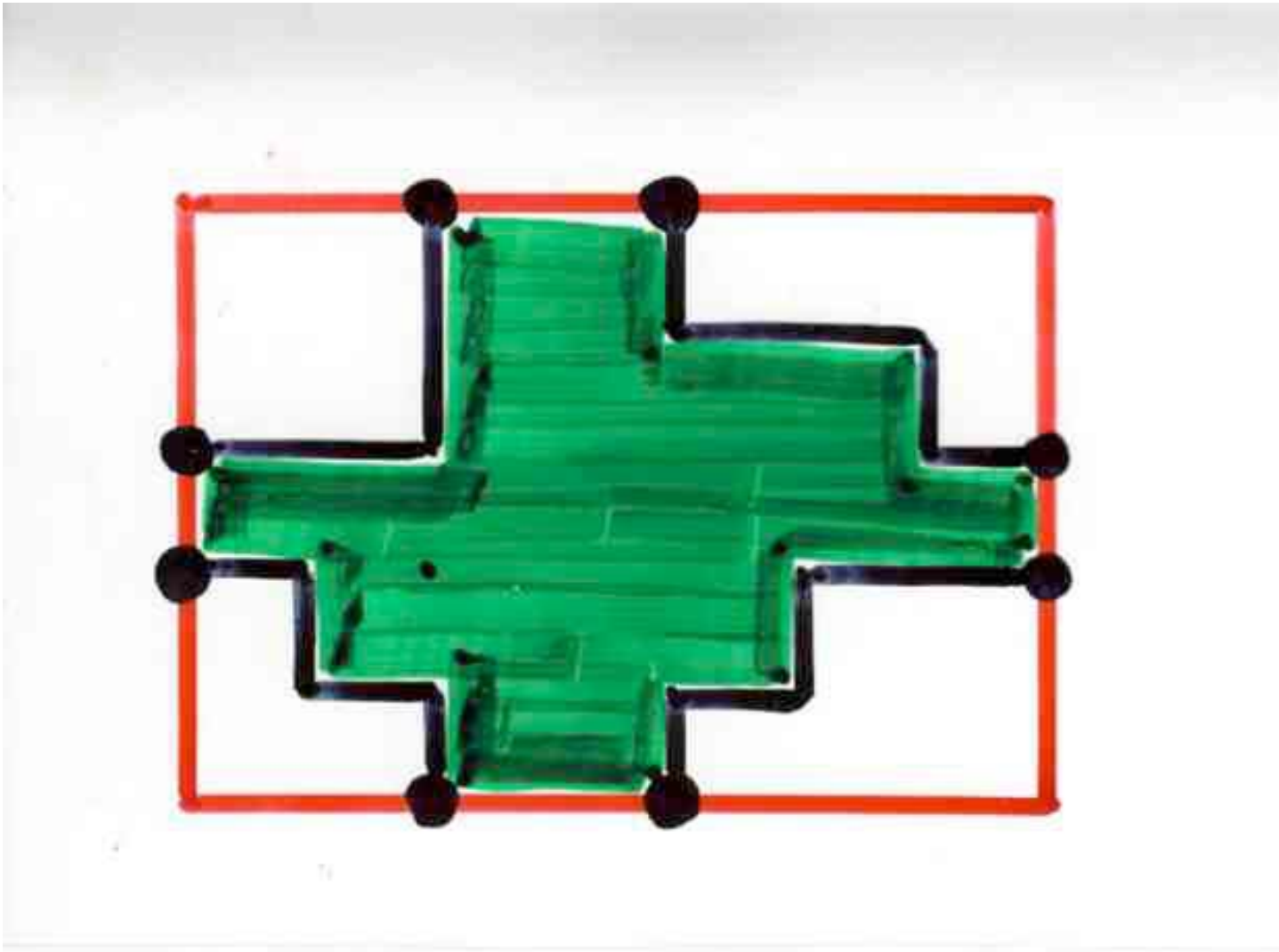
b_m = number of polyominoes with **perimeter** m

© Ministère de l'Éducation nationale

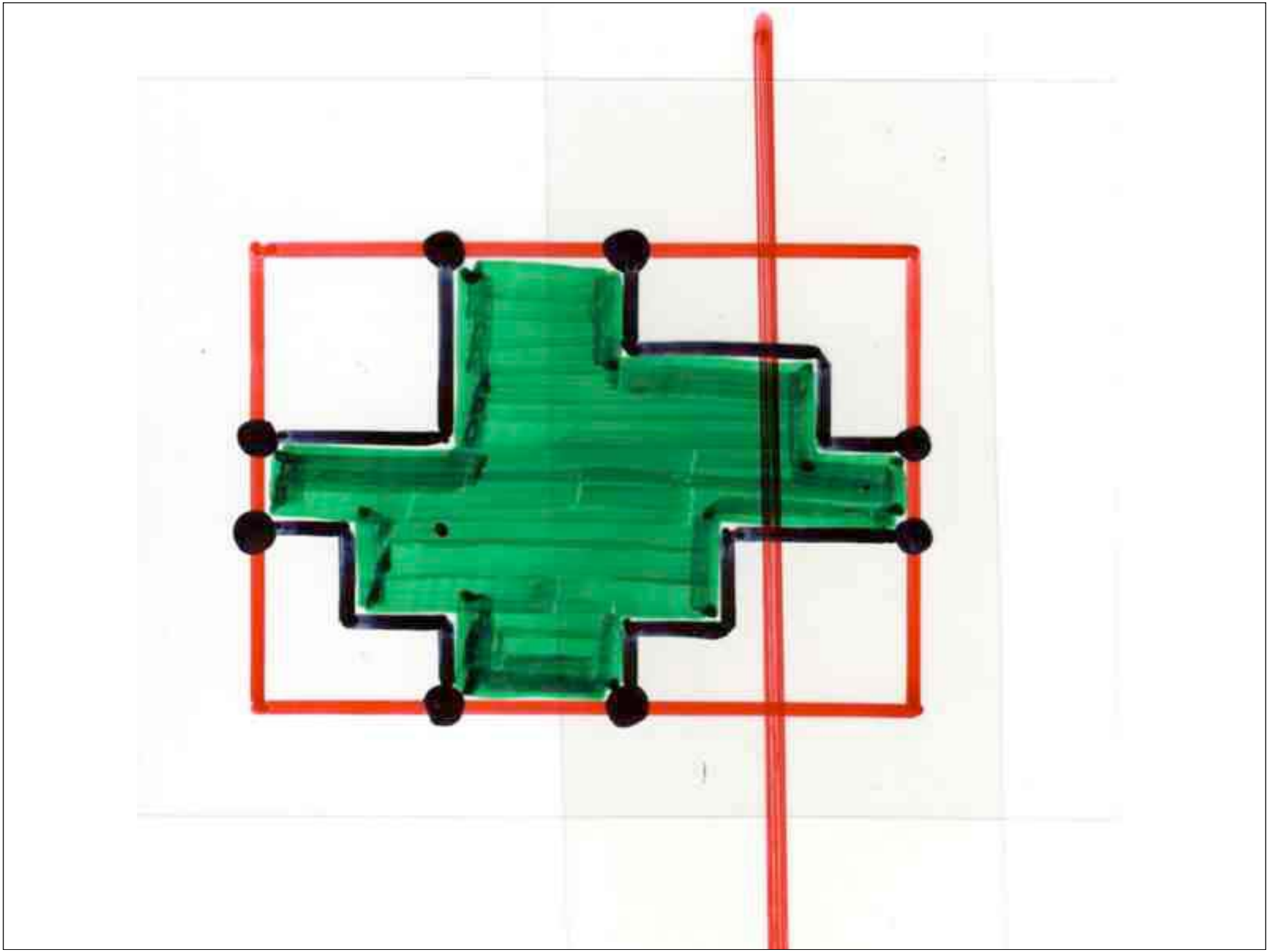
© Ministère de l'Éducation nationale



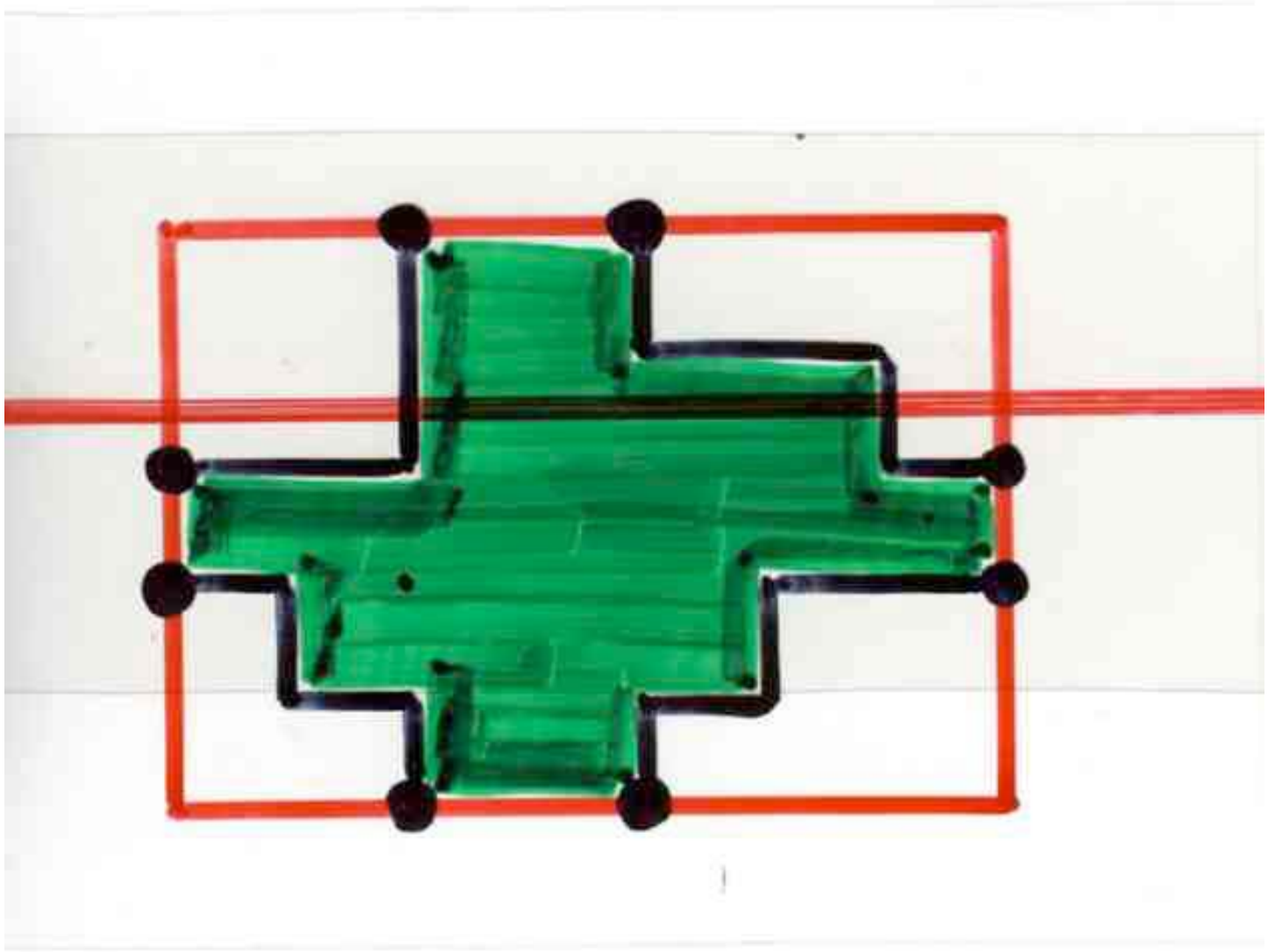
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

J. Phys. A: Math. Gen. 21 (1988) L467–L474. Printed in the UK

LETTER TO THE EDITOR

The number of convex polygons on the square and honeycomb lattices

A J Guttmann† and I G Enting‡

† Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia
‡ CSIRO Division of Atmospheric Research, Private Bag 1, Mordialloc, Victoria 3195, Australia

Received 23 October 1987

Abstract. A subset of the set of self-avoiding polygons (SAP) embeddable on the square lattice which display the property of convexity is defined. An algorithm for their enumeration is developed, and from the available series coefficients the exact generating function is found. The singularity structure appears similar to that of the unsolved SAP problem, but with different critical exponents and critical points.

The enumeration of convex polygons allows the extension of the existing series for the square lattice SAP by one term. For the honeycomb lattice similar results have been obtained, despite a less natural definition of convexity.

Direction générale de l'Enseignement scolaire
© Ministère de l'Éducation nationale

$$R_{ij}^{k+1} = \sum_{m=i}^{j-1} \sum_{n=m+1}^{j} R_{mn}^{k} \qquad 0 < i < j < N$$

$$S_{0j}^{k+1} = \sum_{n=1}^{j} S_{0n}^{k} + \sum_{m=1}^{j-1} \sum_{n=m+1}^{j} R_{mn}^{k} \qquad 0 < j < N$$

$$S_{ij}^{k+1} = \sum_{m=0}^{i} \sum_{n=i+1}^{j} S_{mn}^{k} \qquad 0 < i < j < N$$

$$T_{iN}^{k+1} = \sum_{m=i}^{N-1} T_{mN}^{k} + \sum_{m=i}^{N-2} \sum_{n=m+1}^{N-1} R_{mn}^{k} \qquad 0 < i < N$$

$$T_{ij}^{k+1} = \sum_{m=i}^{j-1} \sum_{n=j}^{N} T_{mn}^{k} \qquad 0 < i < j < N$$

$$U_{iN}^{k+1} = \sum_{m=0}^{i} U_{mN}^{k} + \sum_{m=0}^{i} \sum_{n=i+1}^{N-1} S_{mn}^{k} \qquad 0 < i < N$$

© Ministère de l'Éducation nationale

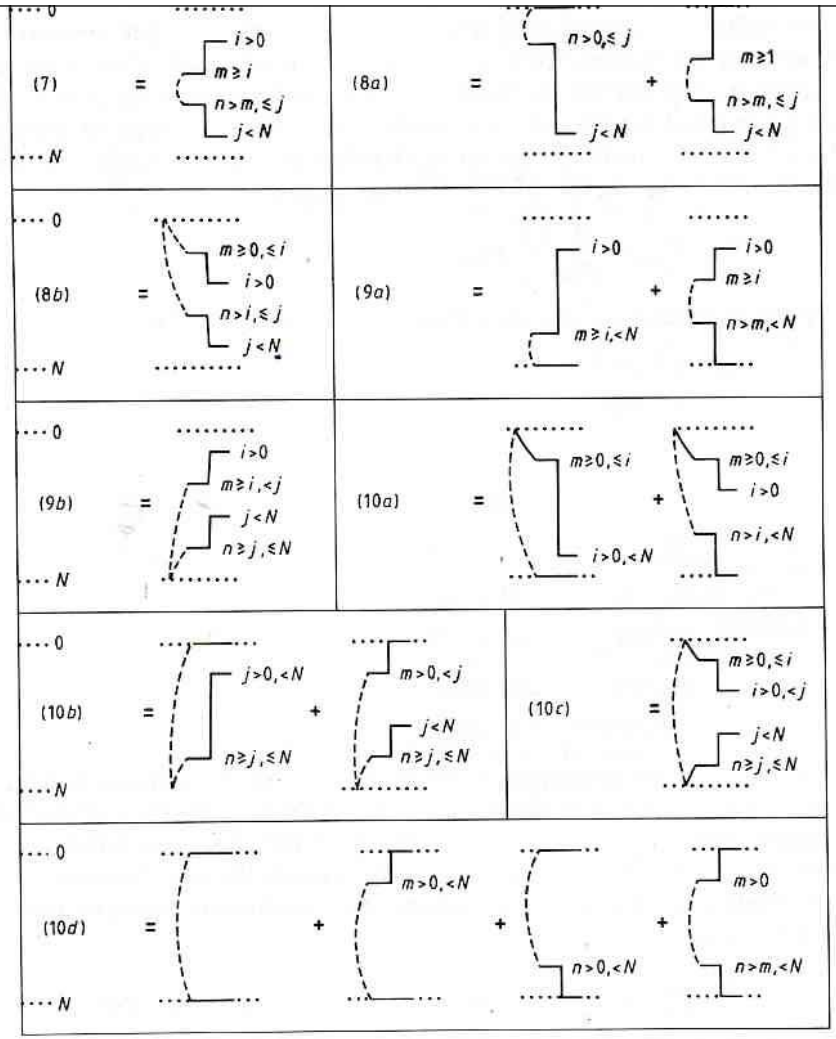


Figure 1. Graphical representation of the transfer matrix equations for generating convex polygons of width N. Each diagram represents one of the single or double summations in the specified equation (7)-(10), with the ordering of diagrams following the order of terms in the equation. The boundaries O and N are shown as dotted lines. The diagrams show the evolution from column k (left) to column $k+1$ (right). The broken curve shows (schematically) the connection to the left of column k and whether these connecting bonds pass through the O and/or N boundaries. The horizontal bonds are indexed i, j, m, n as in the equations and the limits on these indices (which define the limits on the summations) are shown. The index on the left of each inequality is the index of the bond opposite which the inequality is shown. Two-sided bounds ($a<j<b$) on a bond j are shown as $j<a, <b$.

© Ministère de l'Éducation nationale

Table 1. The number of convex polygons of N steps emb honeycomb lattices.

N	Square	Honeycomb
4	1	0
6	2	1
8	7	0
10	28	3
12	120	2
14	528	10
16	2 344	14
18	10 416	40
20	46 160	74
22	203 680	176
24	894 312	358
26	3 907 056	798
28	16 986 352	1 670
30	73 512 288	3 626
32	316 786 960	7 638
34	1 359 763 168	16 366
36	5 815 457 184	34 462
38	24 788 842 304	73 230
40	105 340 982 248	153 830
42	446 389 242 480	324 896
44	1 886 695 382 192	680 514
46	7 955 156 287 456	1 430 336
48	33 468 262 290 096	2 987 310
50	140 516 110 684 832	6 253 712
52	588 832 418 973 280	13 025 954
54	2 463 133 441 338 048	27 176 052
56	10 286 493 304 041 104	56 465 878
58	42 892 130 604 098 656	117 458 820
60	178 592 047 539 343 200	243 507 250
62	742 609 229 473 744 320	505 239 624
64	2 439 630 075 430 725 288	

© Ministère de l'Éducation nationale

We define the generating function as

$$P(x) = x^{-2} \sum_{n=2}^{\infty} P_{2n} x^n \tag{11}$$

where the term $1/x^2$ takes into account the fact that the first non-zero coefficient is p_4. By a systematic search, we find that the coefficients p_{2n} satisy the recurrence relation

$$\begin{aligned}[(n+1)^2 - 9.5(n+1) + 15]p_{2n+6} - (8n^2 - 58n + 57)p_{2n+4} \\ + [16(n-1)^2 - 80(n-1) + 36]p_{2n+2} \\ = 7\delta_{n,2} + 22\delta_{n,1} - 44\delta_{n,0} + 15\delta_{n,-1}.\end{aligned} \tag{12}$$

The class of recurrence relations within which the search was made is inspired by the exact results for the two-dimensional Ising model magnetisation and specific heat. We then provide a check on the recurrence relation. This recurrence relation among the coefficients is equivalent to the following differential equation for the generating function:

$$\begin{aligned}P''(x)(x^2 - 8x^3 + 16x^4) - P'(x)(8.5x - 50x^2 + 64x^3) + P(x)(15 - 57x + 36x^2) \\ = 15 - 44x + 22x^2 - 7x^3.\end{aligned} \tag{13}$$

© Ministère de l'Éducation nationale

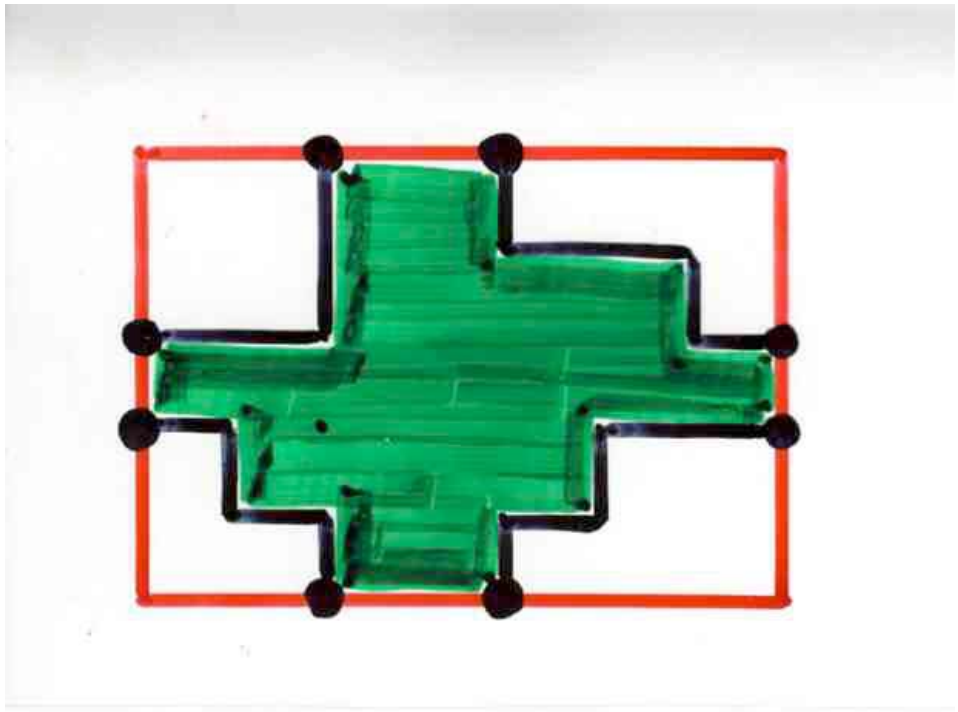
it follows that $c=0$ (or else powers of $u^{1/2}$ would be present). Integration of (3.7) and substitution of the early terms of the series to identify the constant of integration yields the solution

$$P(x)=(1-6x+11x^2-4x^3)/(1-4x)^2-4x^2/(1-4x)^{3/2}. \tag{18}$$

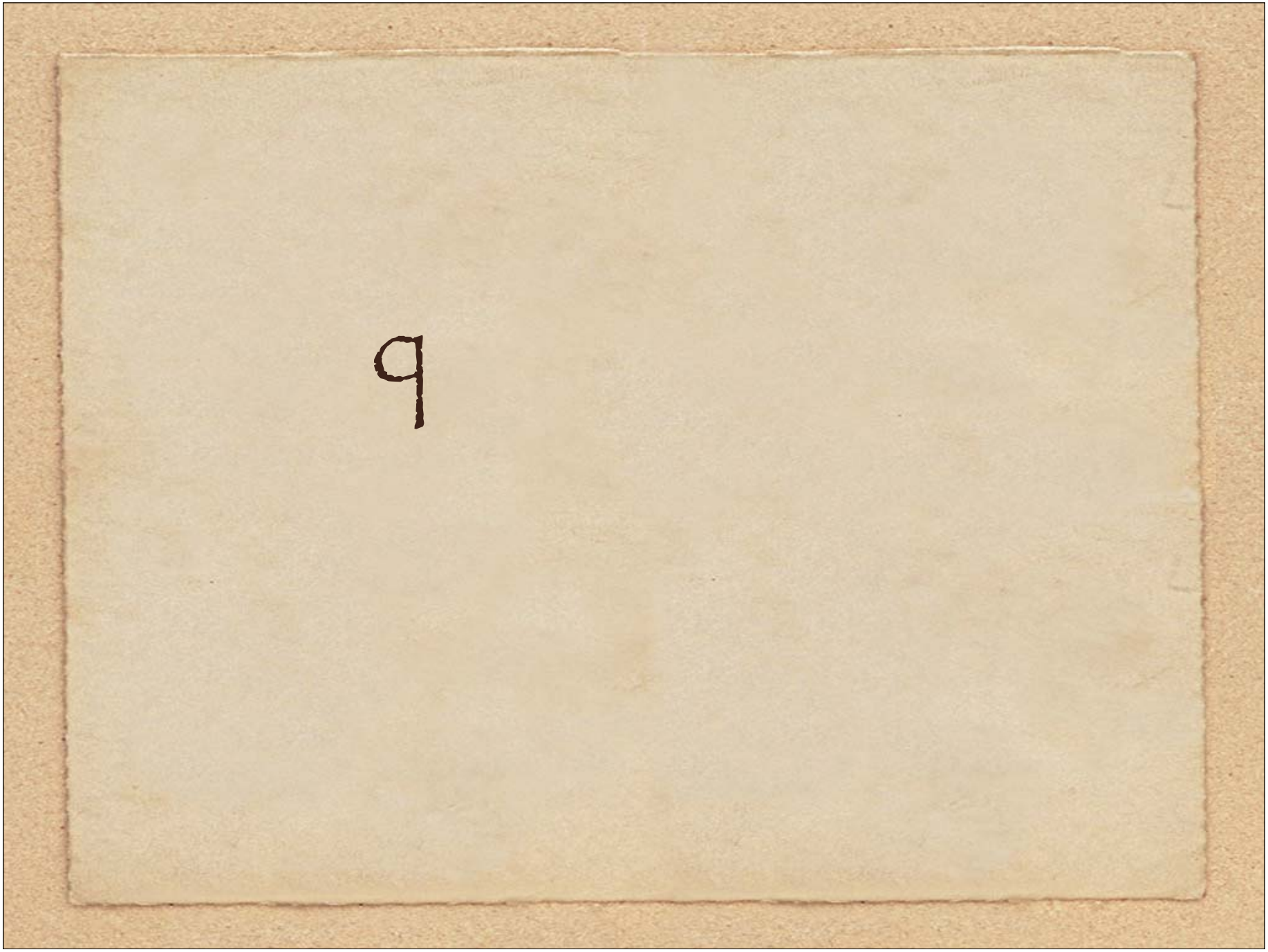
That is, the generating function has a 'critical point' at $x=\frac{1}{4}$, with critical exponent 2

© Ministère de l'Éducation nationale

$$P_{2n+3} = (2n+11)\,2^{2n} - 4(n+1)\binom{2n}{n}$$



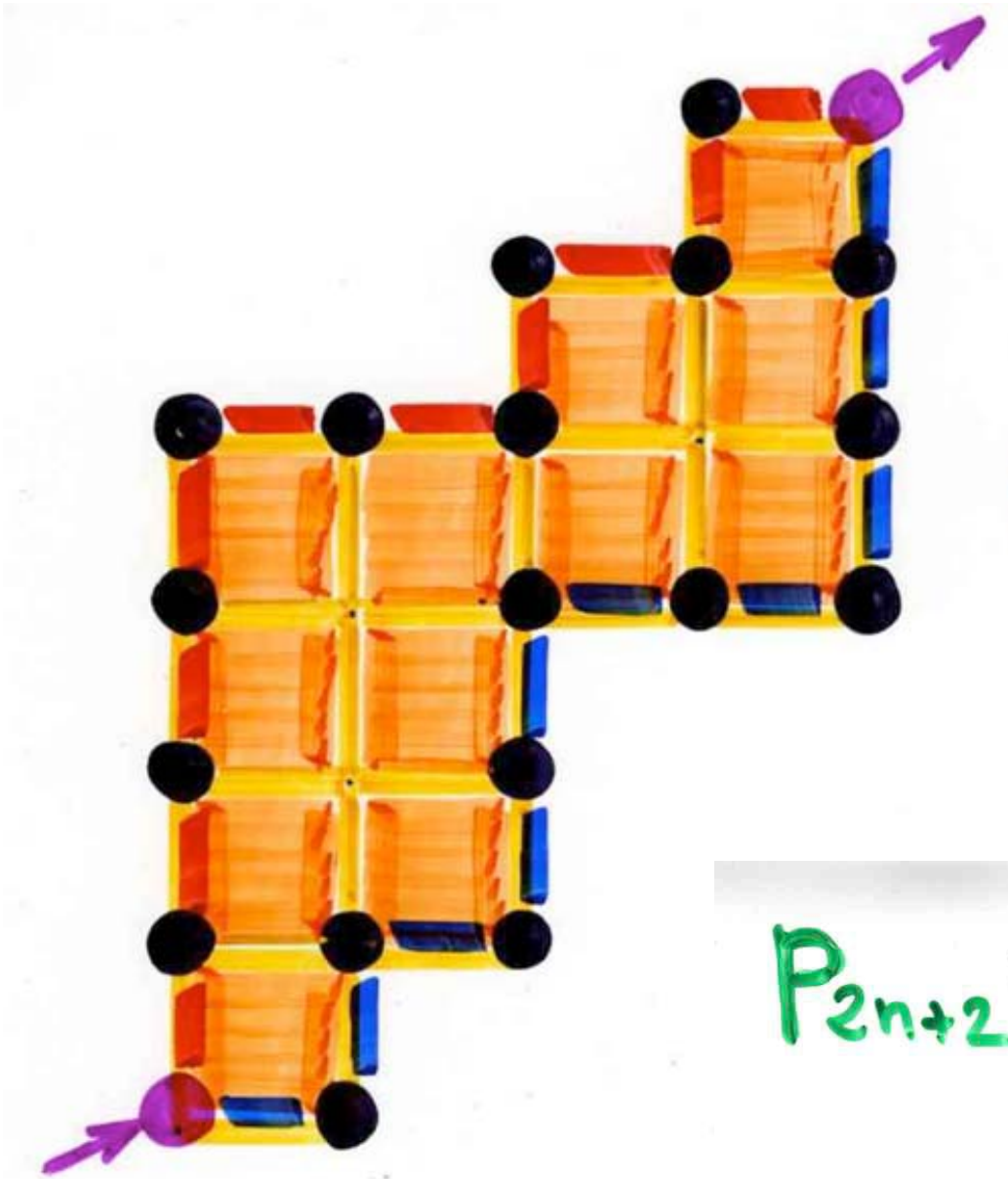
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

Polyomino
parallélogramme

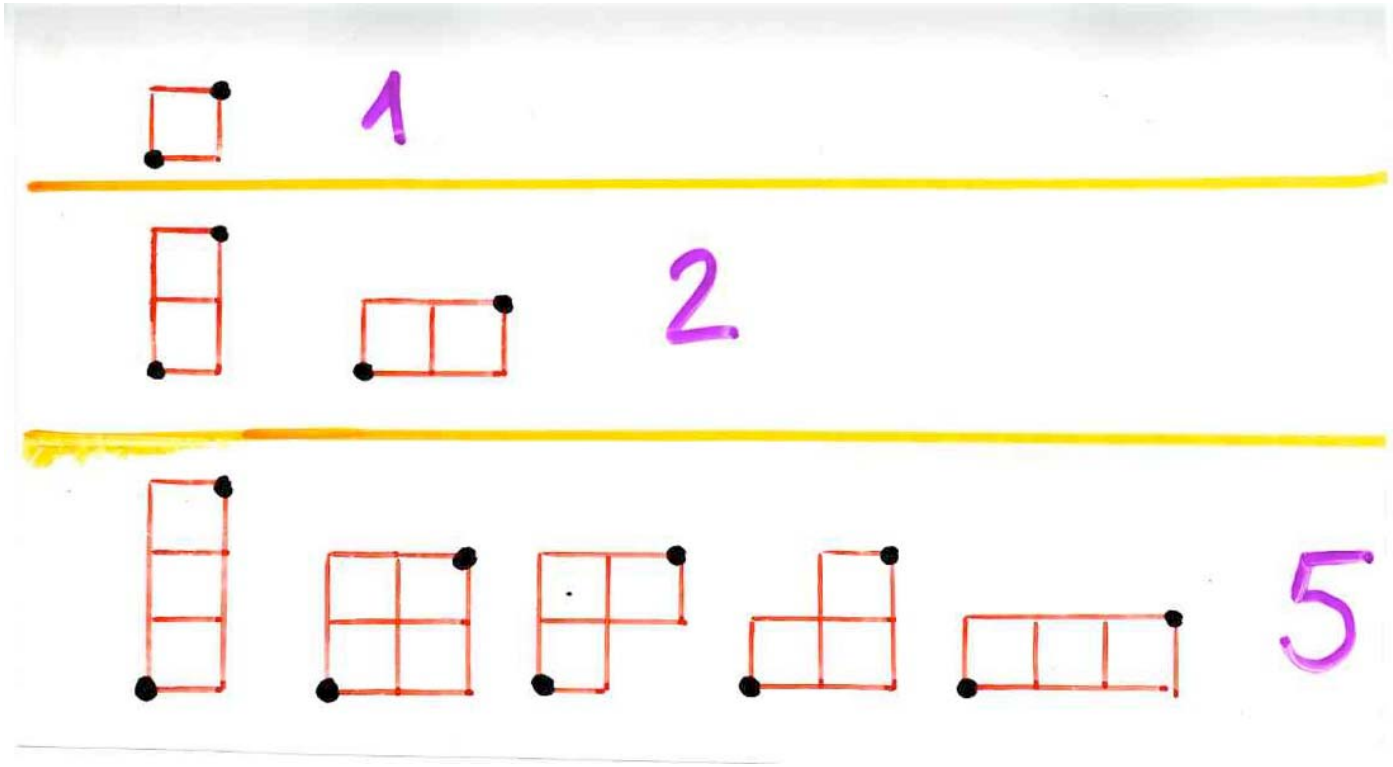
© Ministère de l'Éducation nationale



Polyomino
parallélogramme

$$P_{2n+2} = \frac{1}{n+1}\binom{2n}{n}$$

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

$f(t,q)$ generating function for parallelogram polyominoes

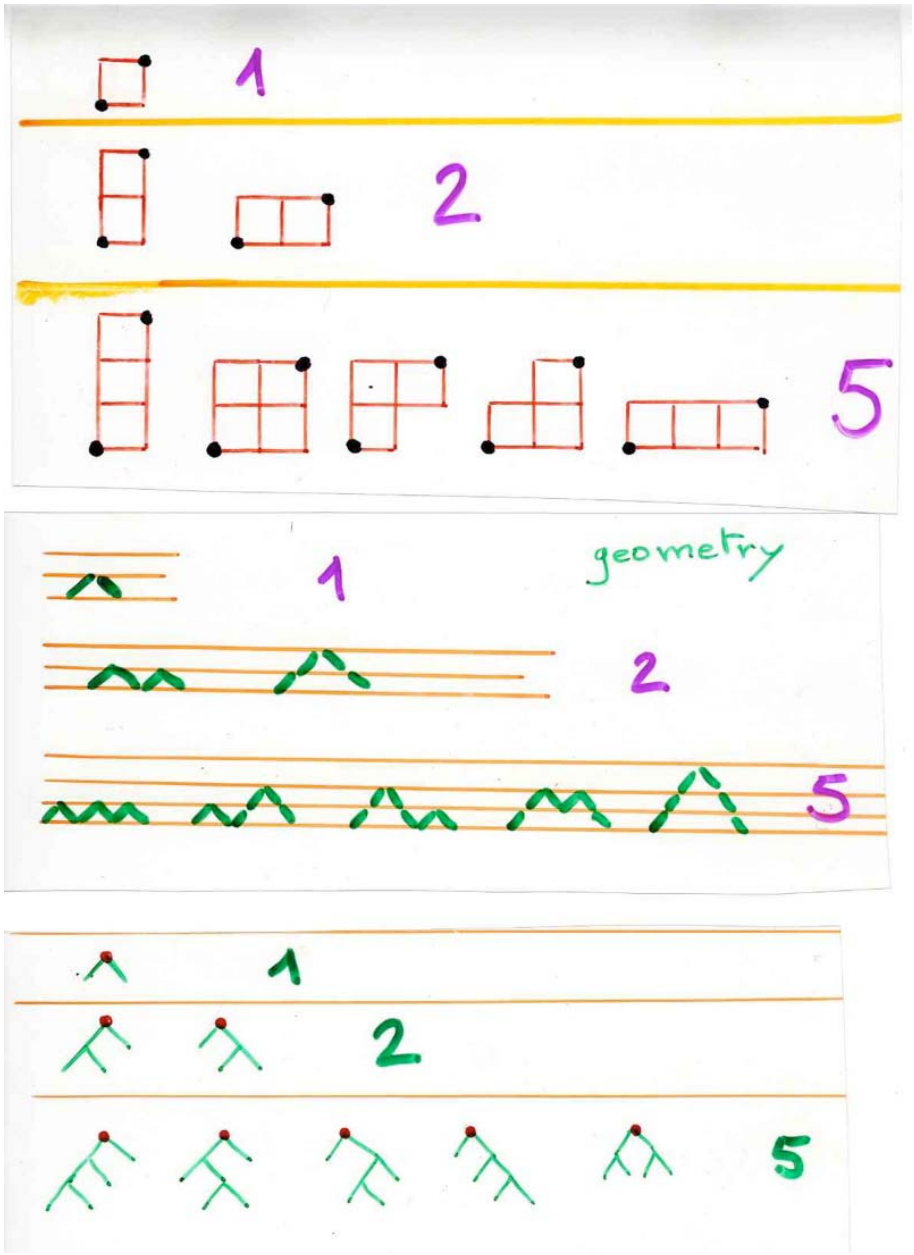
t number of columns

q area

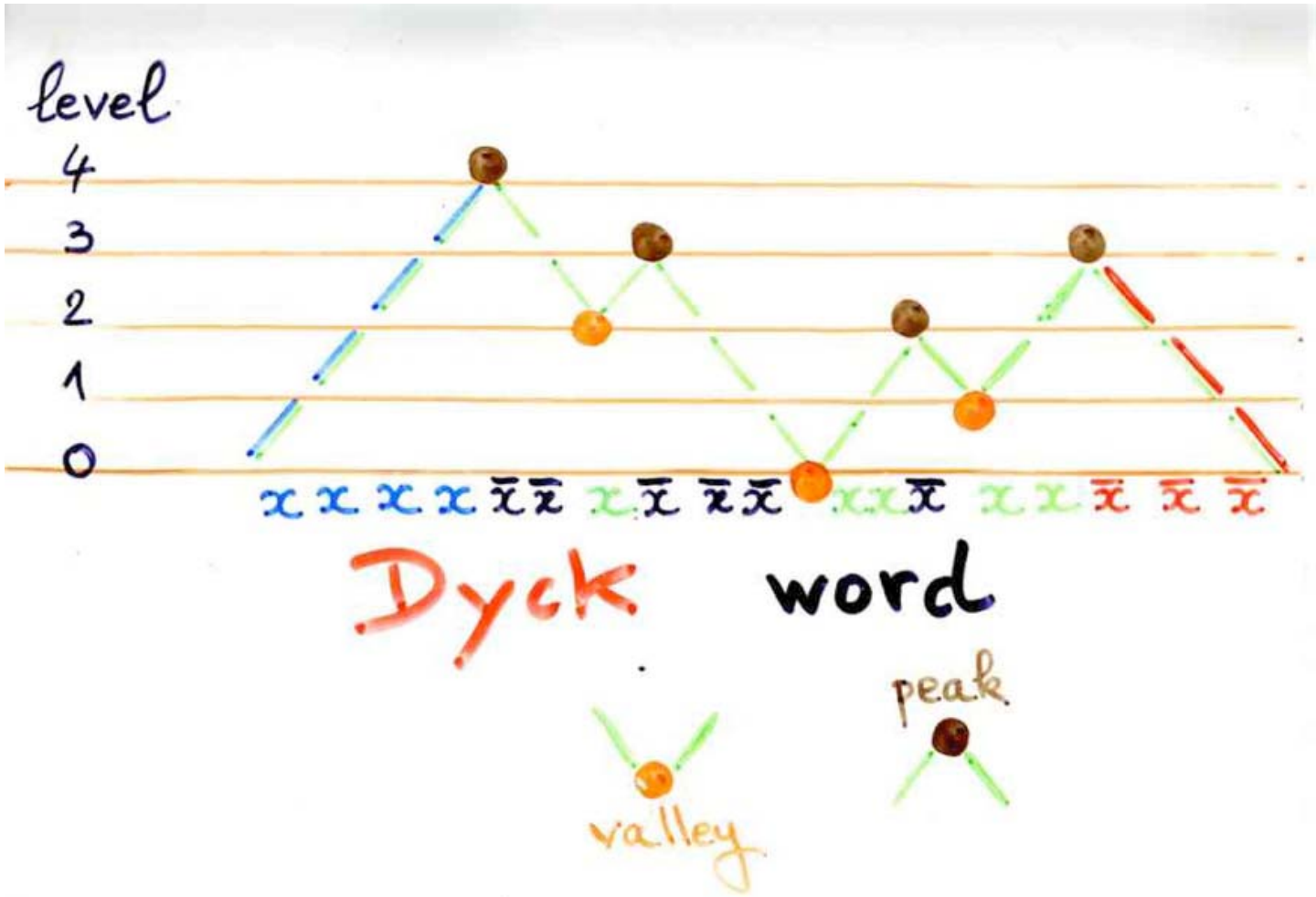
$$f(t,q) = \sum_{n \geq 0} a_n(q)\, t^n$$

rational series

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

$$f(t,q) = 1 + (tq-1) f(t,q) + f(t,q)\, f(tq,q)$$

q- equation différentielle

Fedou, Delest

© Ministère de l'Éducation nationale

$$f(t,q) = \sum_{n \geq 0} a_n(q)\, t^n$$

rational series

© Ministère de l'Éducation nationale

$$a_n(q) = \frac{N_n(q)}{(q;q)_n \, (q;q)_{\lfloor \frac{n}{2} \rfloor} \cdots \; (q;q)_{\lfloor \frac{n}{n} \rfloor}}$$

$$(a;q)_n = (1-a)(1-aq) \ldots (1-aq^{n-1})$$

$N_n(q)$ polynomial

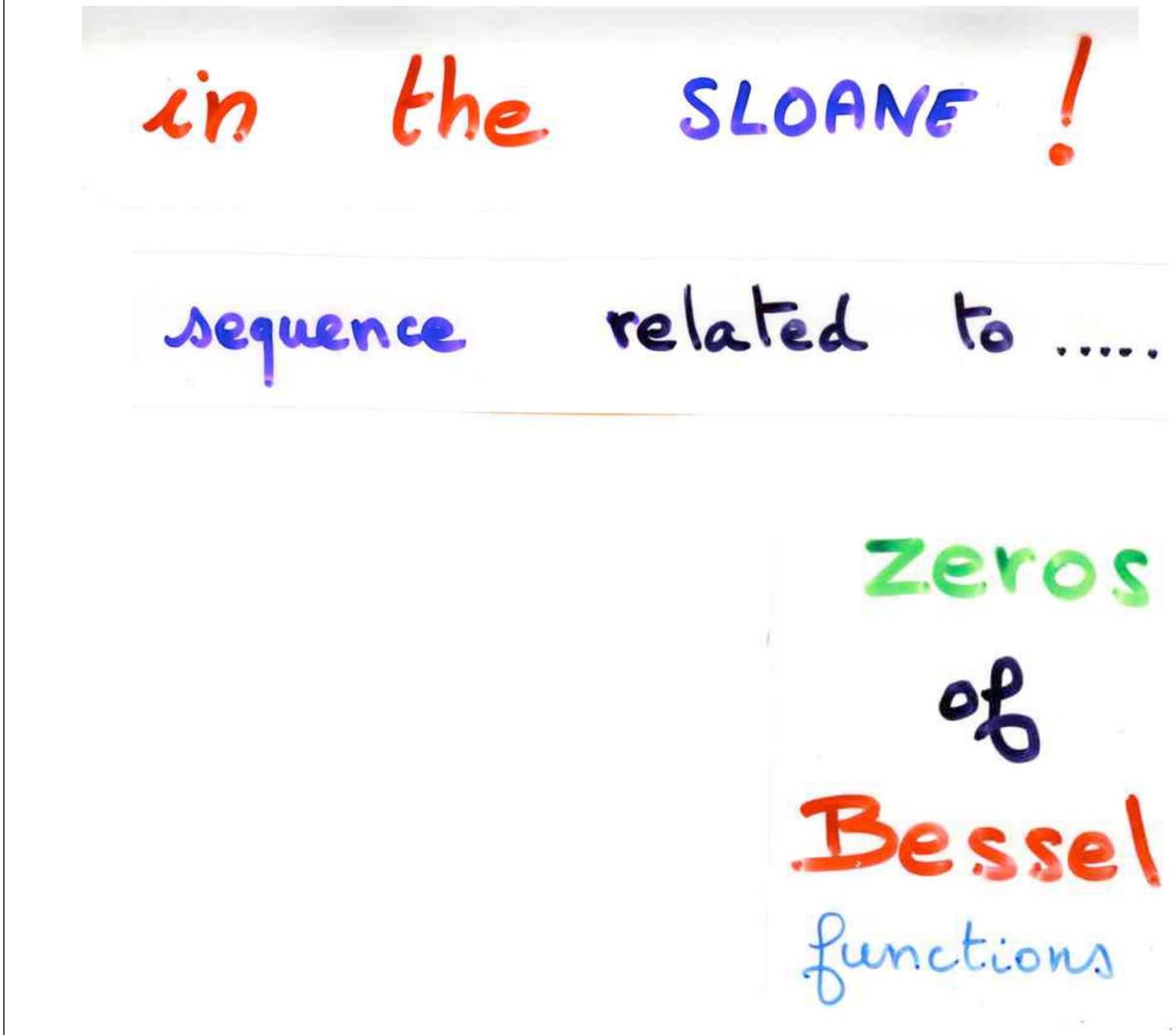
positive integers coefficients

unimodal

$$\alpha_n = N_n(1)$$

$$= 1, 1, 2, 11, 38, 946, \ldots$$

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

Carlitz (1963)

$$J_\nu(t) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{t}{2}\right)^{2n+\nu}}{n!\,(n+\nu)!}$$

$$\frac{J_{\nu+1}(t)}{J_\nu(t)}$$

© Ministère de l'Éducation nationale

guess :

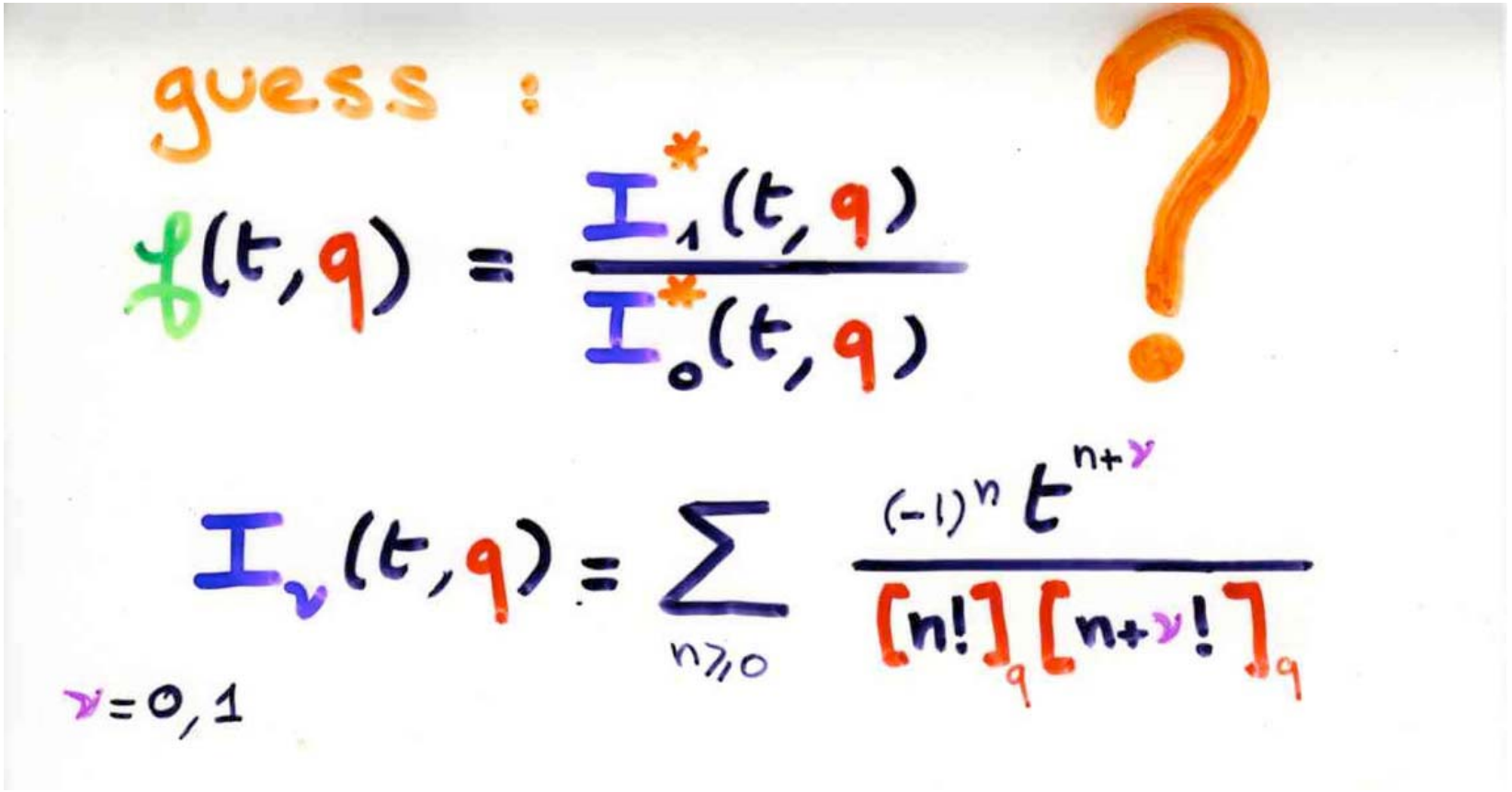
$$f(t,q) = \frac{I_1^*(t,q)}{I_0^*(t,q)}$$

$$I_\nu(t,q) = \sum_{n \geq 0} \frac{(-1)^n t^{n+\nu}}{[n!]_q [n+\nu!]_q}$$

$\nu = 0, 1$

$$(1-q)^n [n]! = (1-q)(1-q^2)\dots(1-q^n)$$
$$= (q;q)_n$$

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

guess :

$$f(t,q) = \frac{I_1^*(t,q)}{I_0^*(t,q)}$$

?

$$I_\nu(t,q) = \sum_{n \geq 0} \frac{(-1)^n t^{n+\nu} q^{n^2}}{[n!]_q [n+\nu!]_q}$$

$\nu = 0, 1$

© Ministère de l'Éducation nationale

guess :

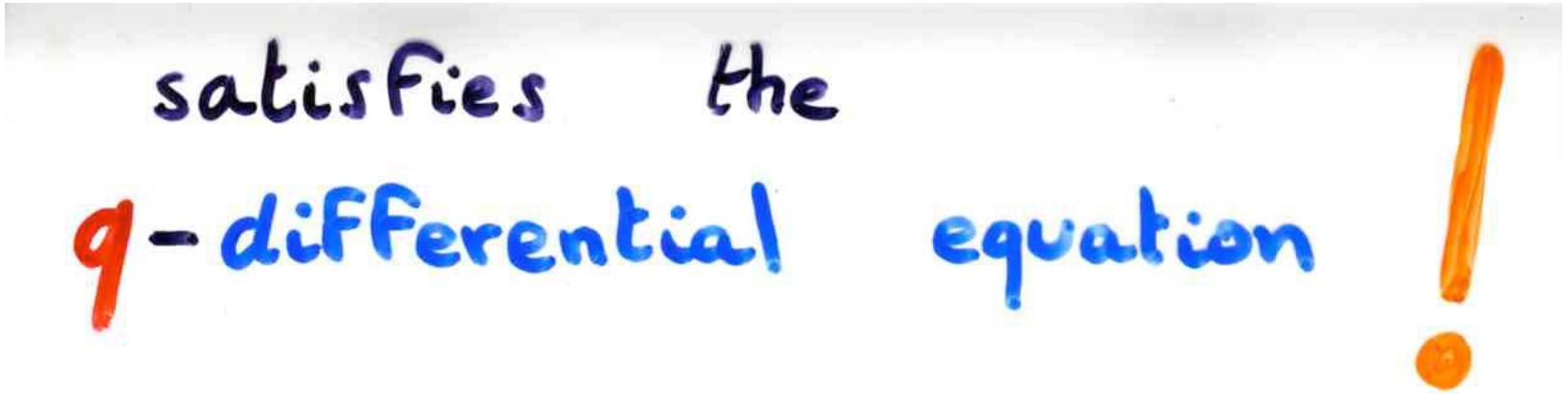
$$f(t,q) = \frac{I_1^*(t,q)}{I_0^*(t,q)}$$

?

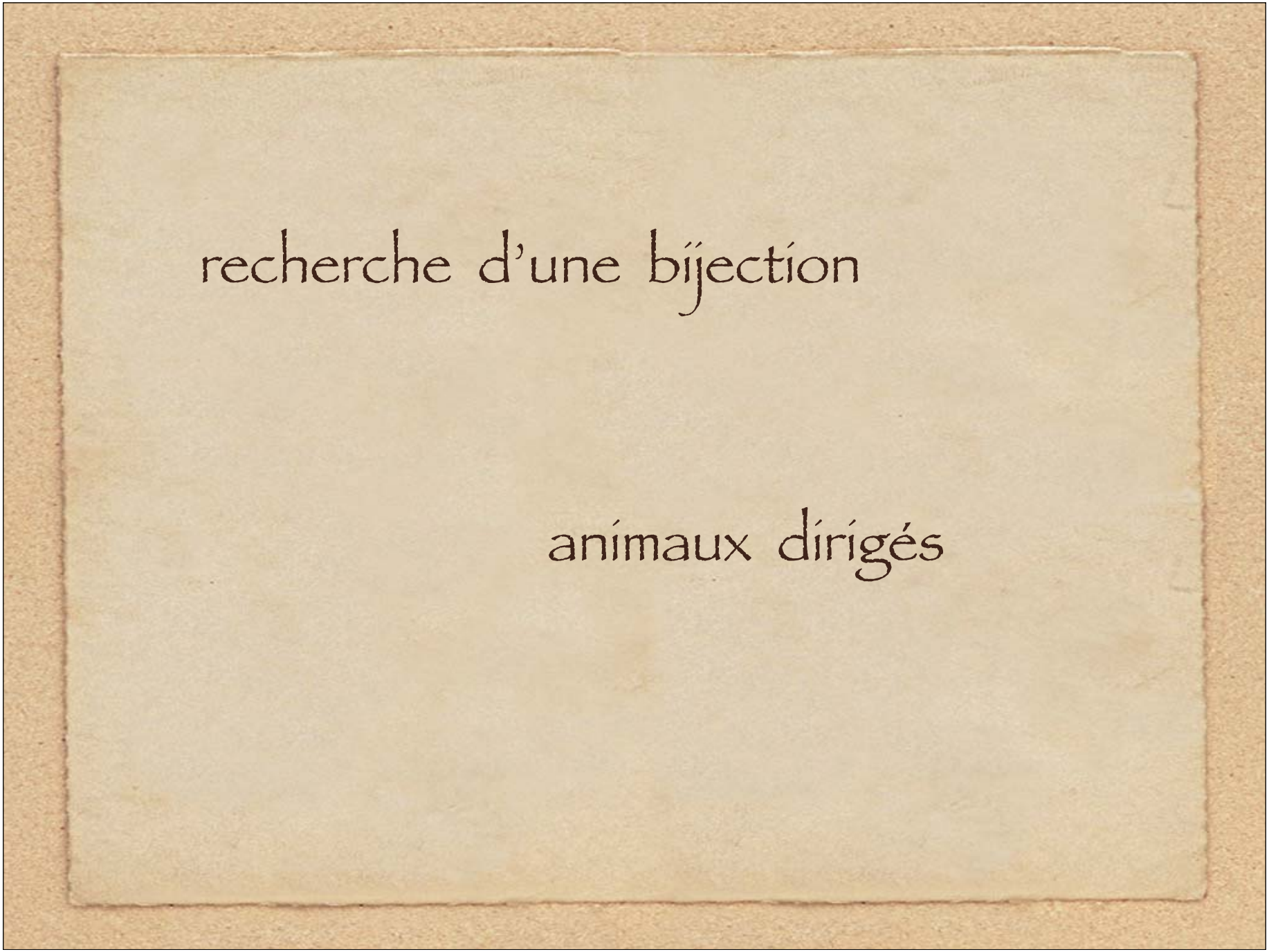
$$I_\nu(t,q) = \sum_{n \geq 0} \frac{(-1)^n t^{n+\nu} q^{\binom{n+\nu}{2}}}{[n!]_q [n+\nu!]_q}$$

$\nu = 0, 1$

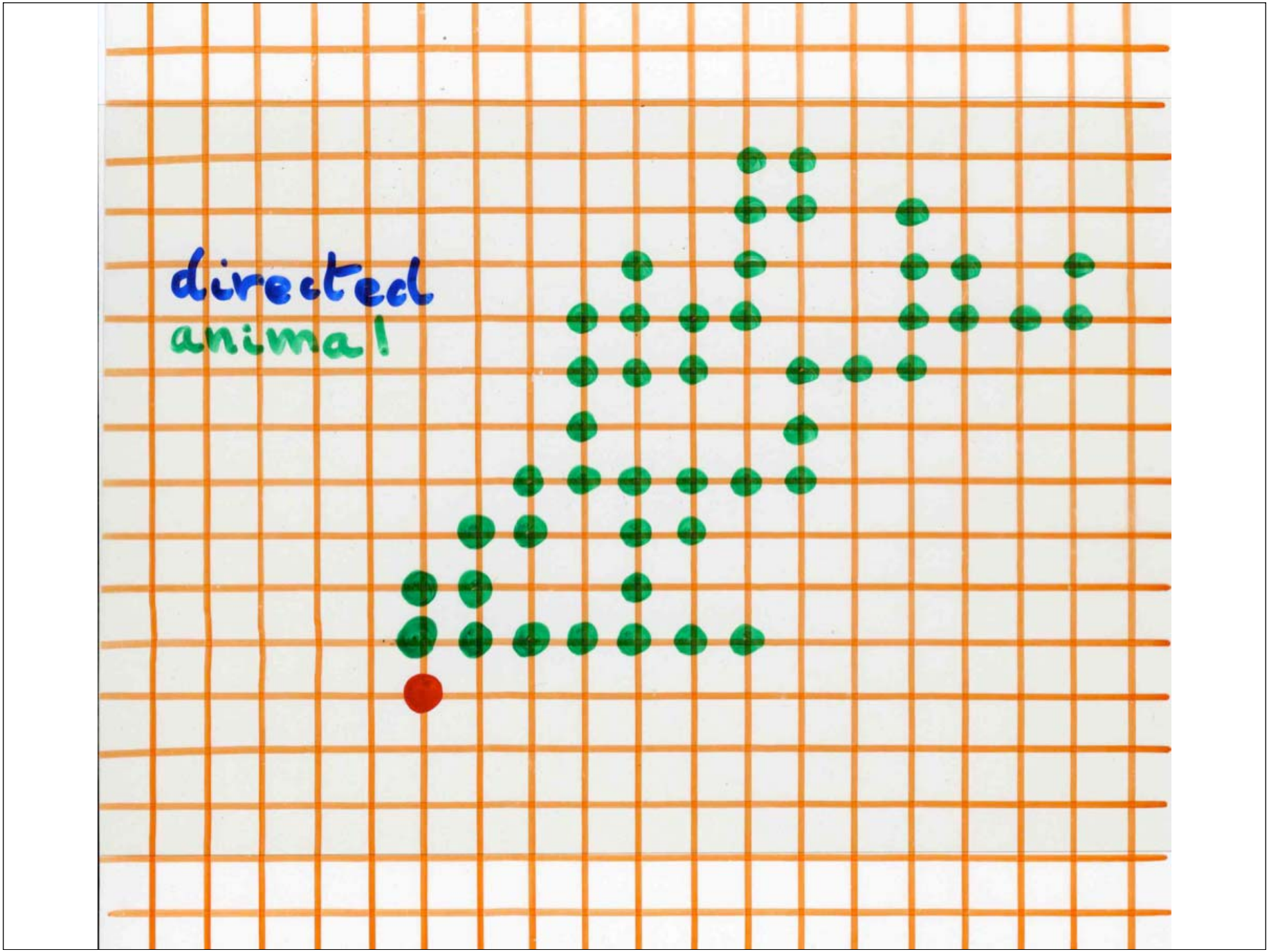
© Ministère de l'Éducation nationale



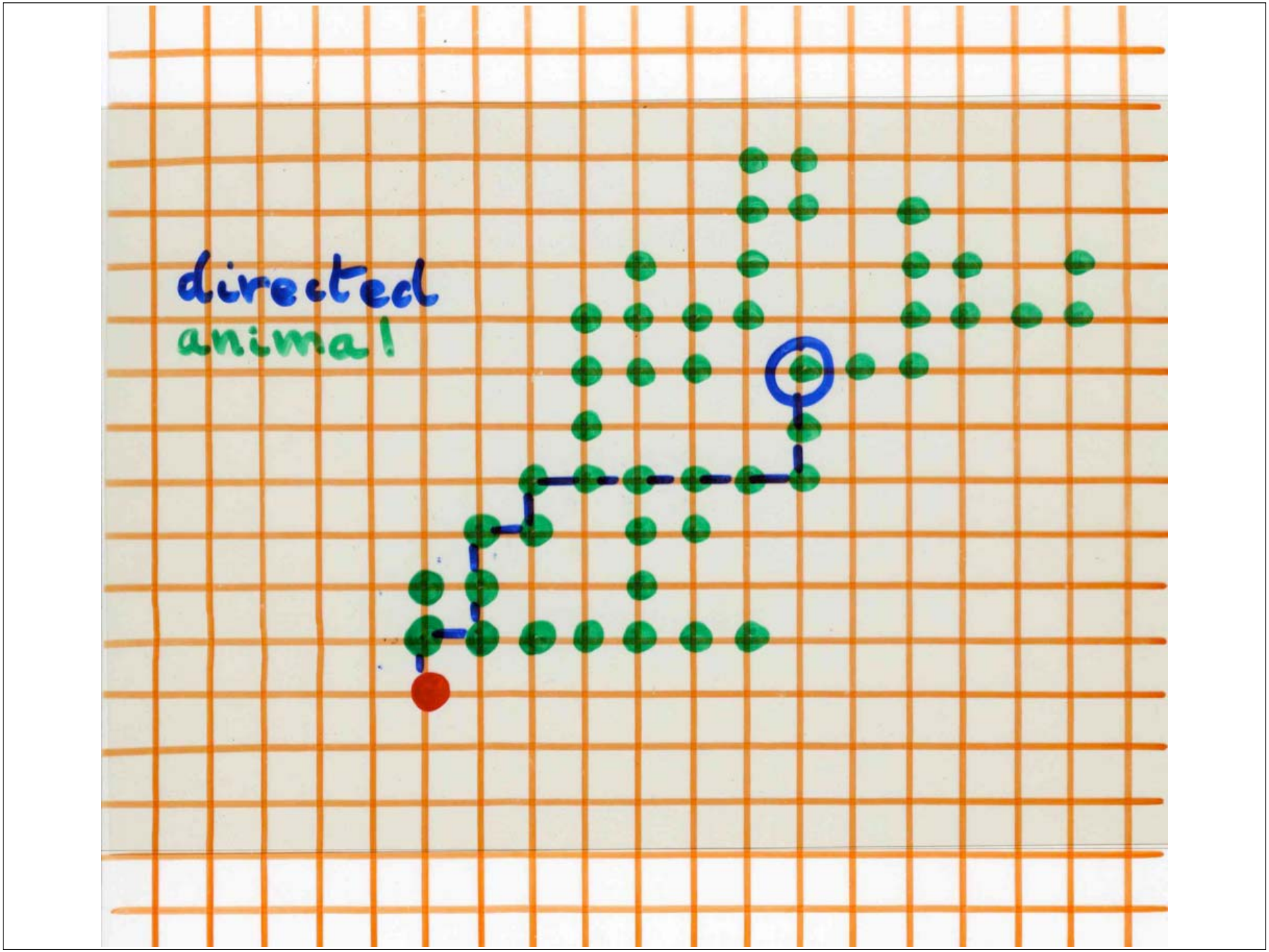
© Ministère de l'Éducation nationale



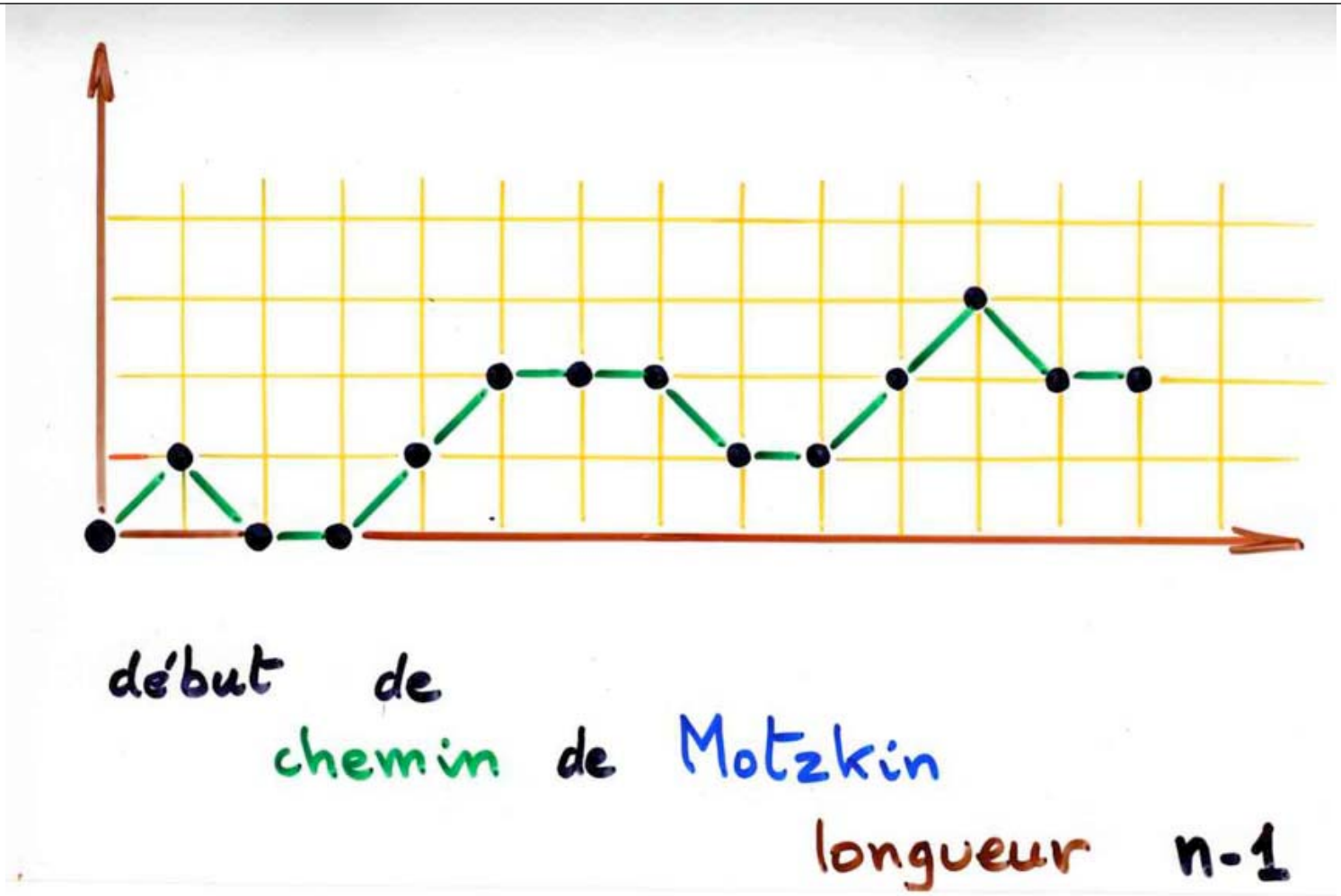
© Ministère de l'Éducation nationale



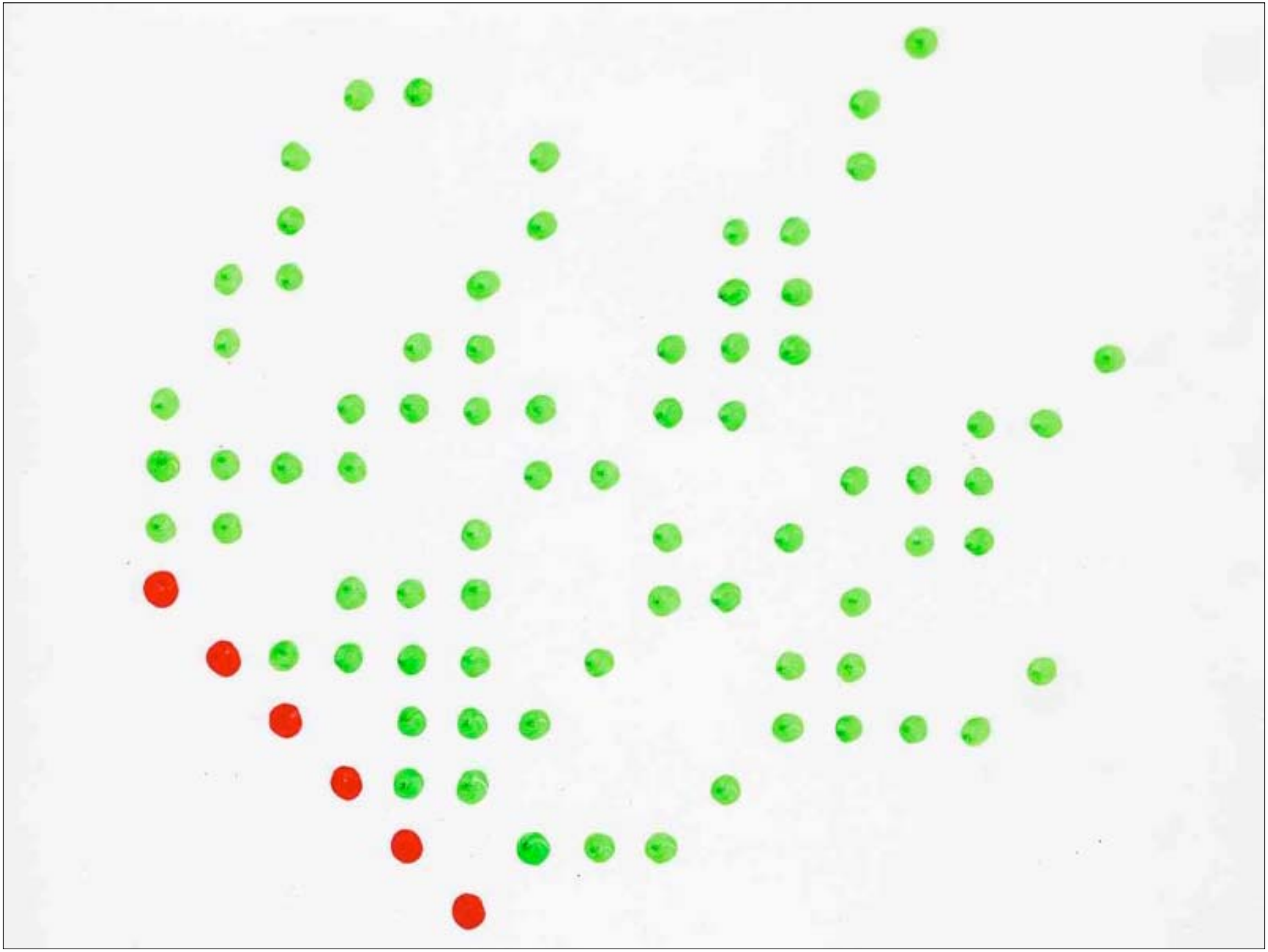
© Ministère de l'Éducation nationale



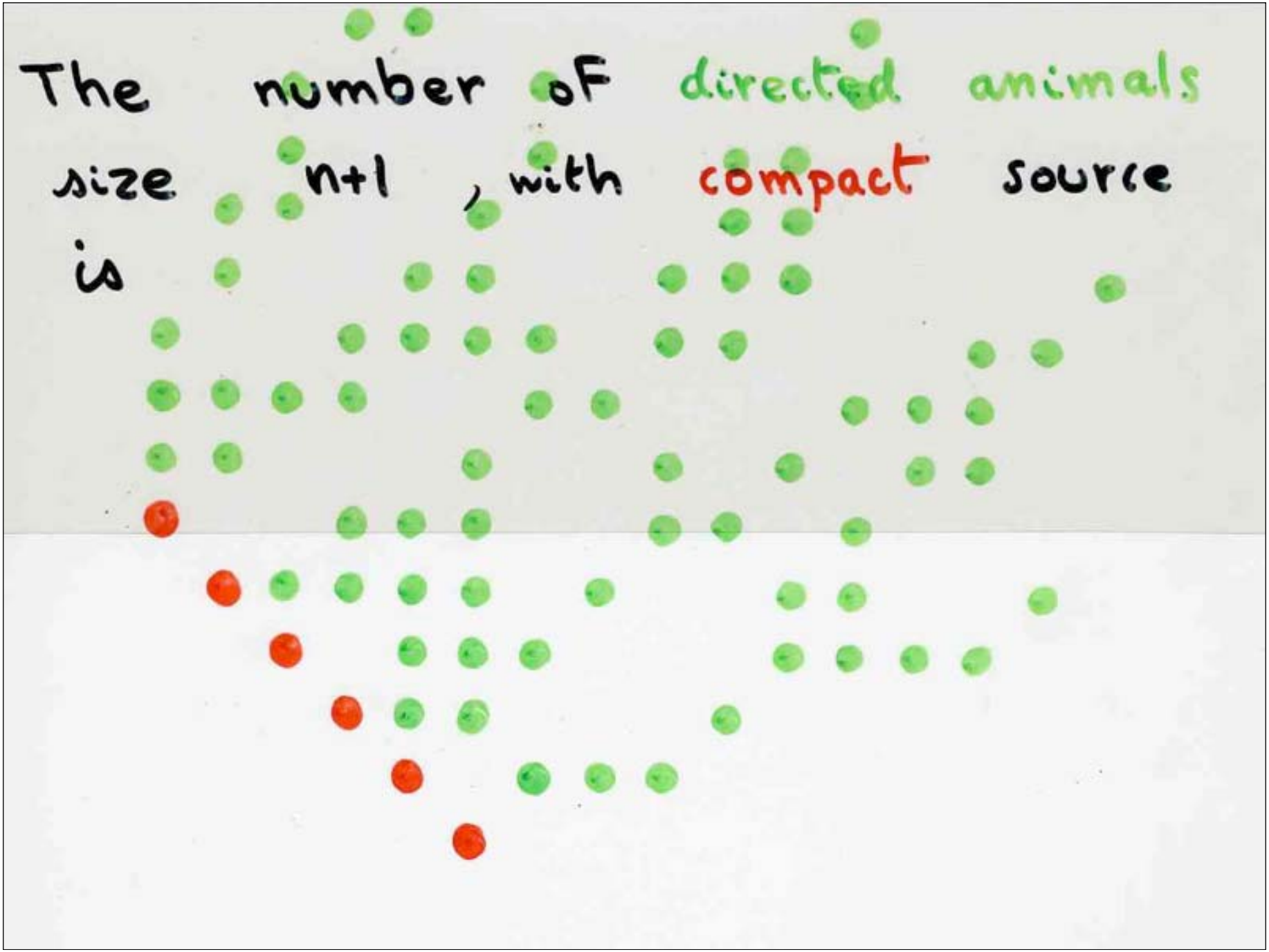
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

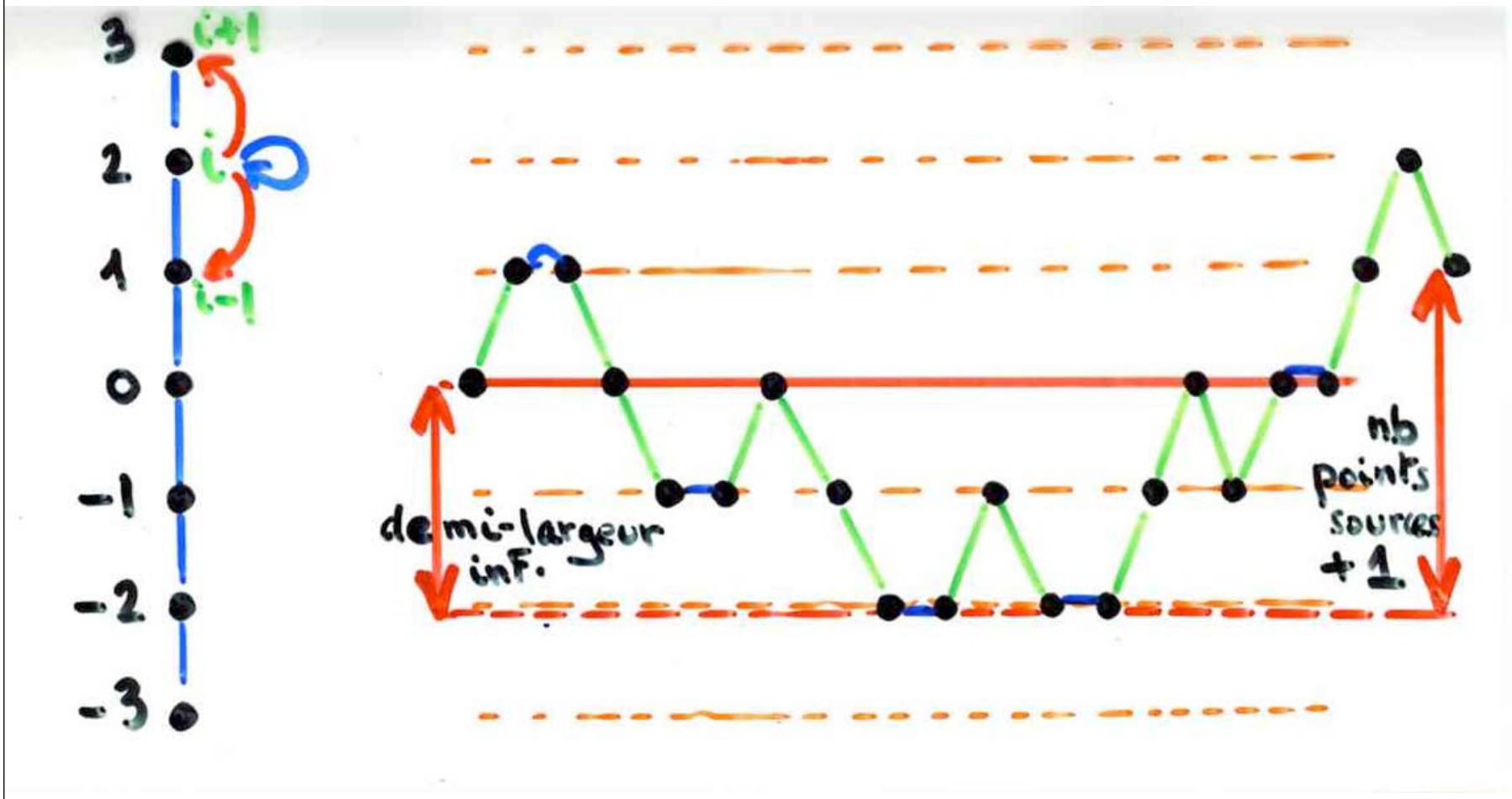


© Ministère de l'Éducation nationale

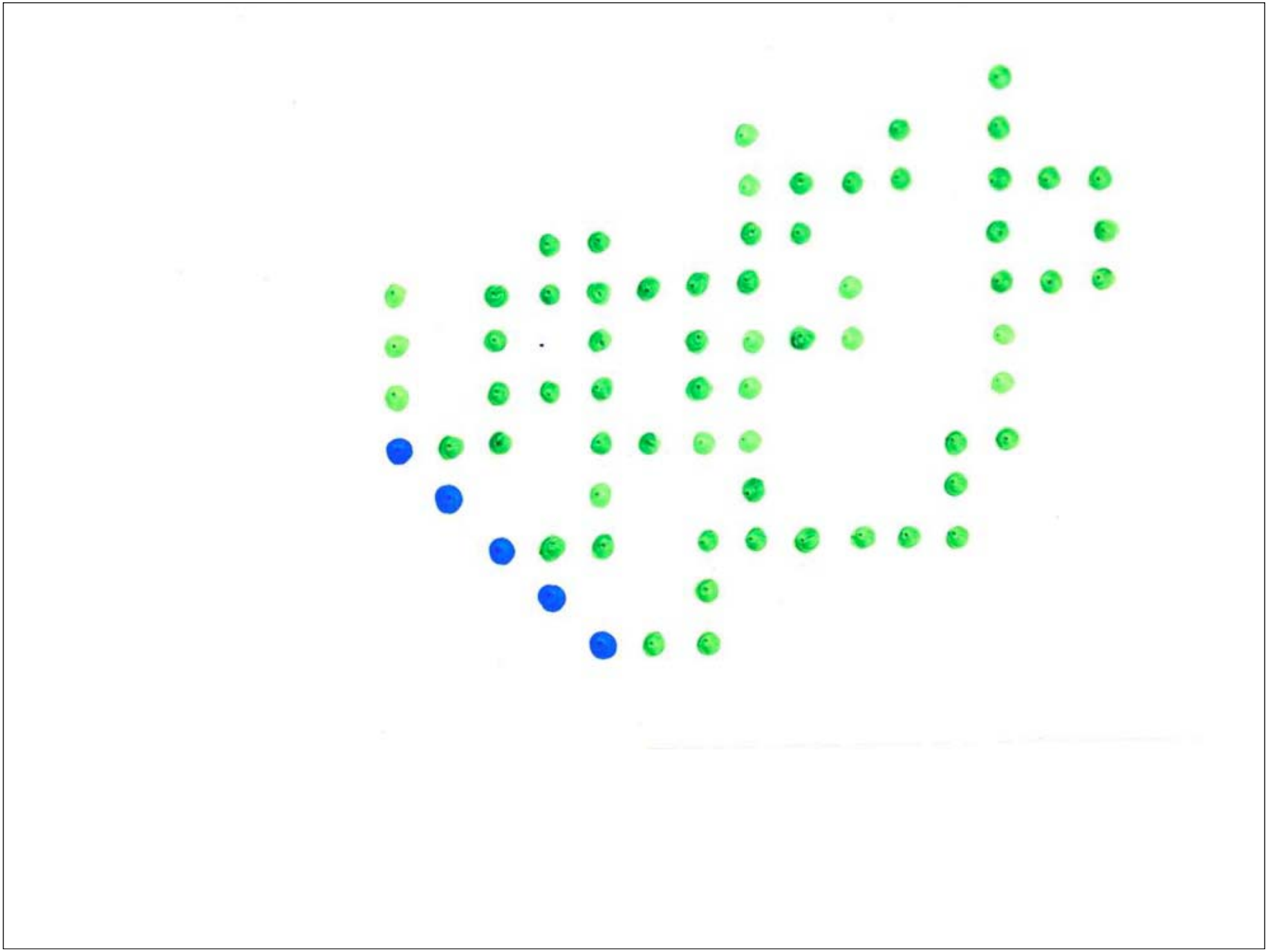
The number of directed animals of size n+1, with compact source is

3^n

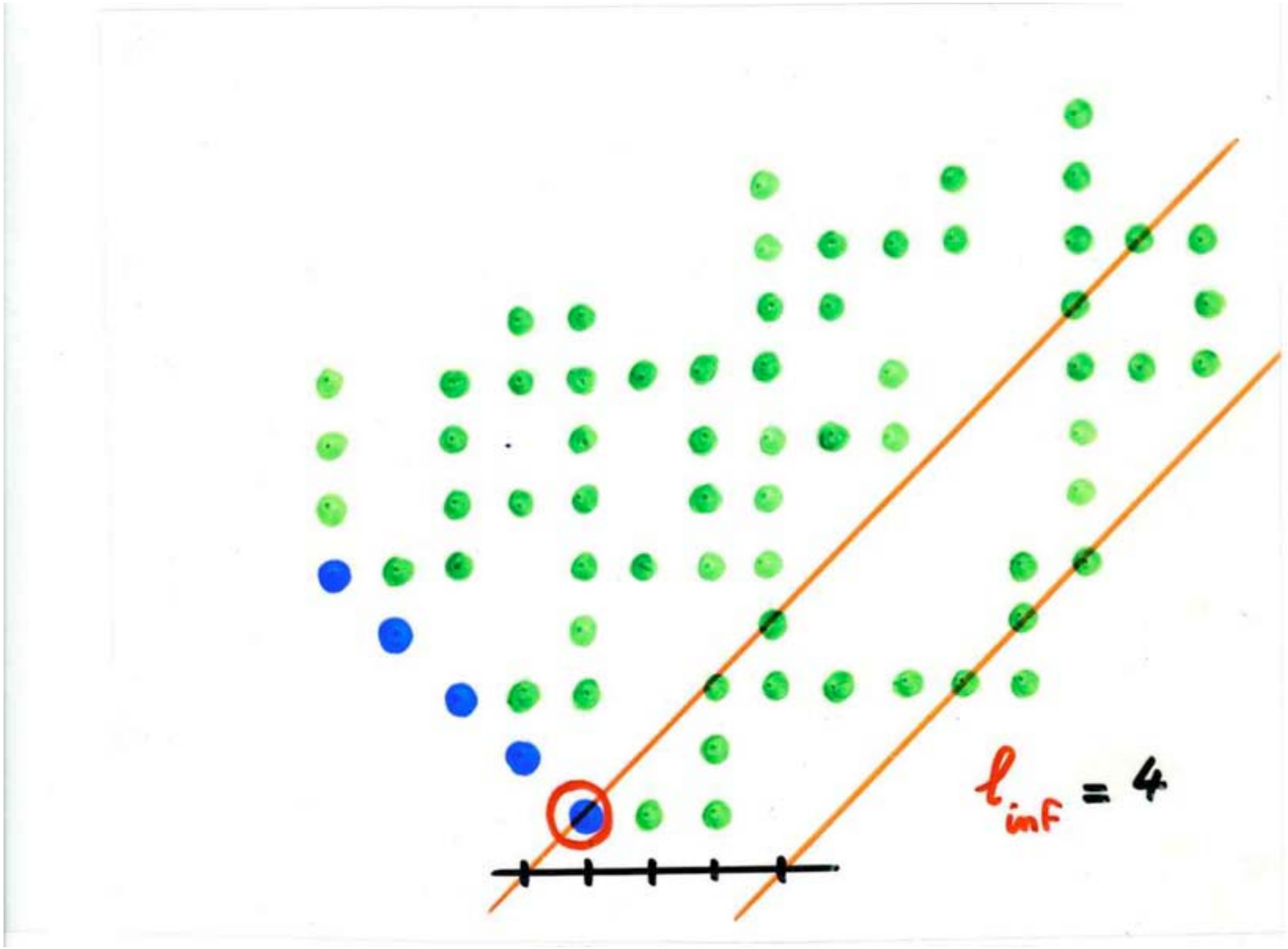
© Ministère de l'Éducation nationale



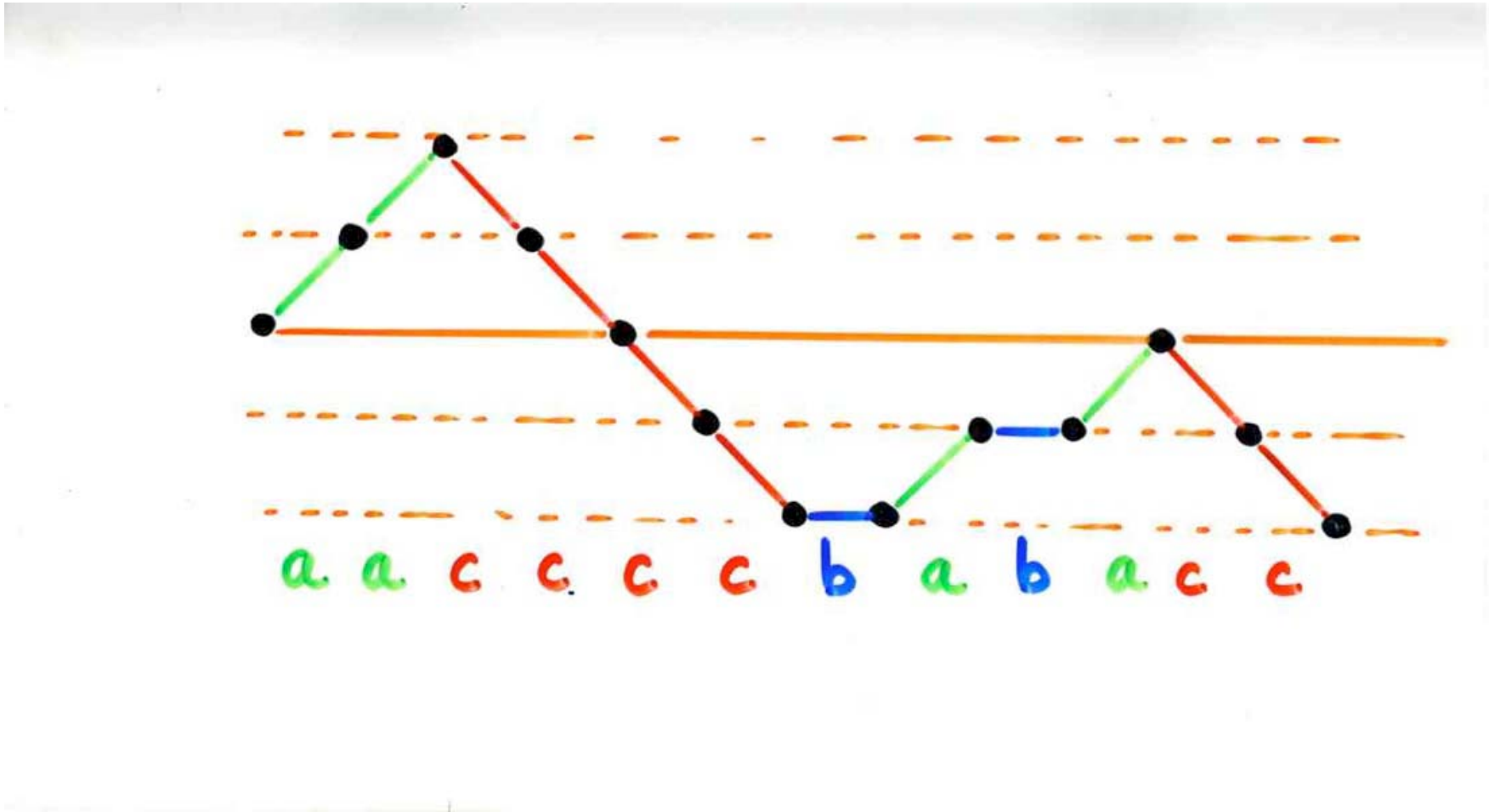
© Ministère de l'Éducation nationale



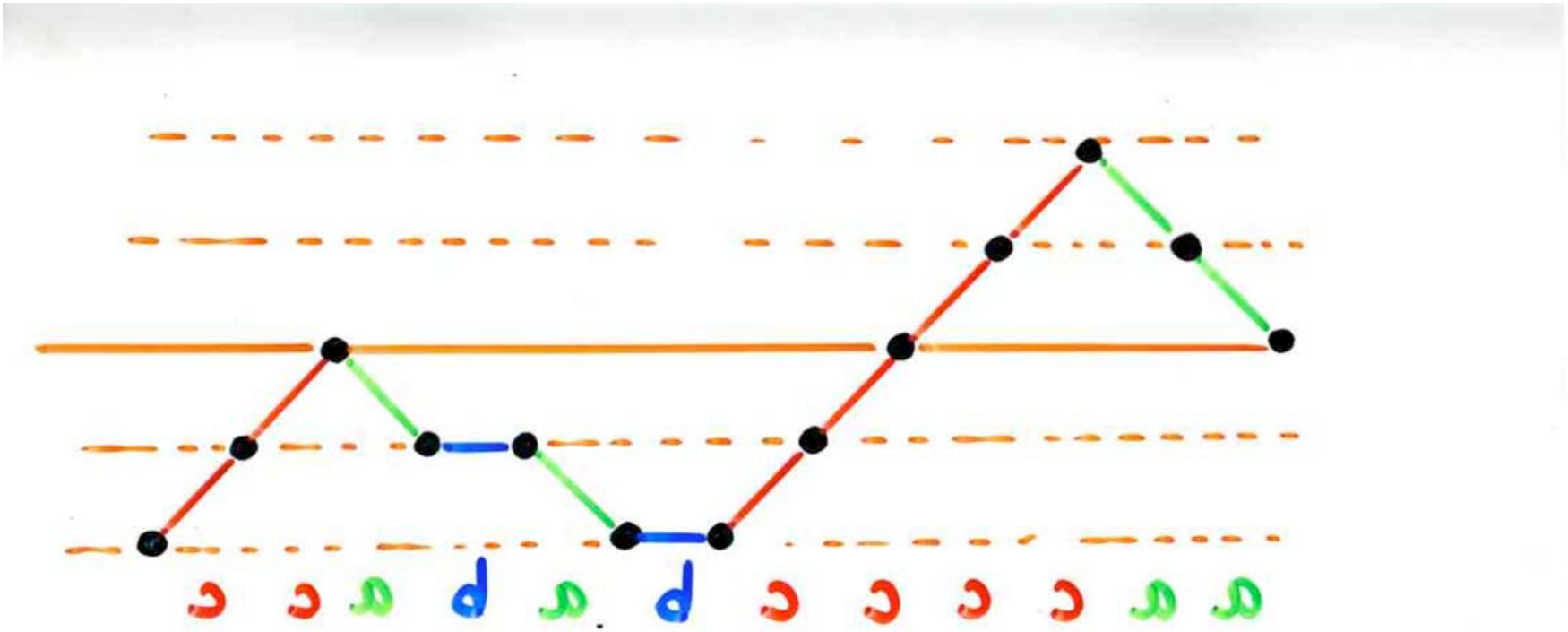
© Ministère de l'Éducation nationale



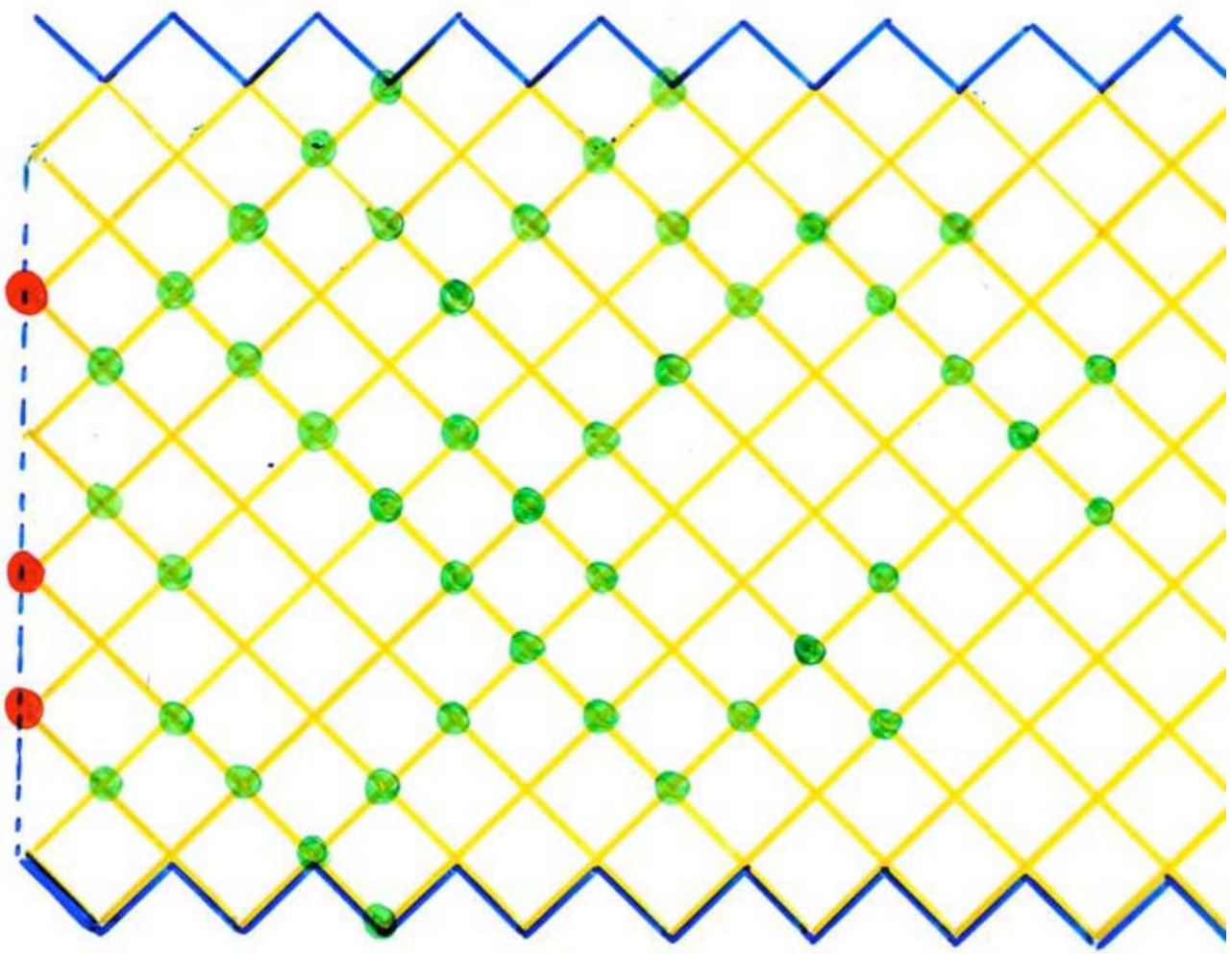
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

Nadal, Derrida, Vannimenus 1982.

Hakim, Nadal 1982.

$$a_n^k \sim c\, \mu_k^n$$

$$\mu_k = 1 + 2\cos \pi/2k$$

$$a_n \sim c\, \mu^n\, n^{-\theta}$$

$$\mu = 3$$

Phenomenological Renormalization

© Ministère de l'Éducation nationale

$$\tilde{b}_n^{\leq k} = \frac{1}{k} \sum_{p=0}^{k-1} (-1)^p \sin \alpha_p \prod_{i=1}^{k-1} \left(\frac{\sin\left(i+\frac{1}{2}\right)\alpha_p}{\sin \frac{\alpha_p}{2}} \right)^{N_i} \left(1+2\cos\alpha_p\right)^{n-1}$$

animals

circular strip

width k

$$\alpha_p = \frac{2p+1}{2k}\pi$$

(Hakim, Nadal, 1982)

© Ministère de l'Éducation nationale

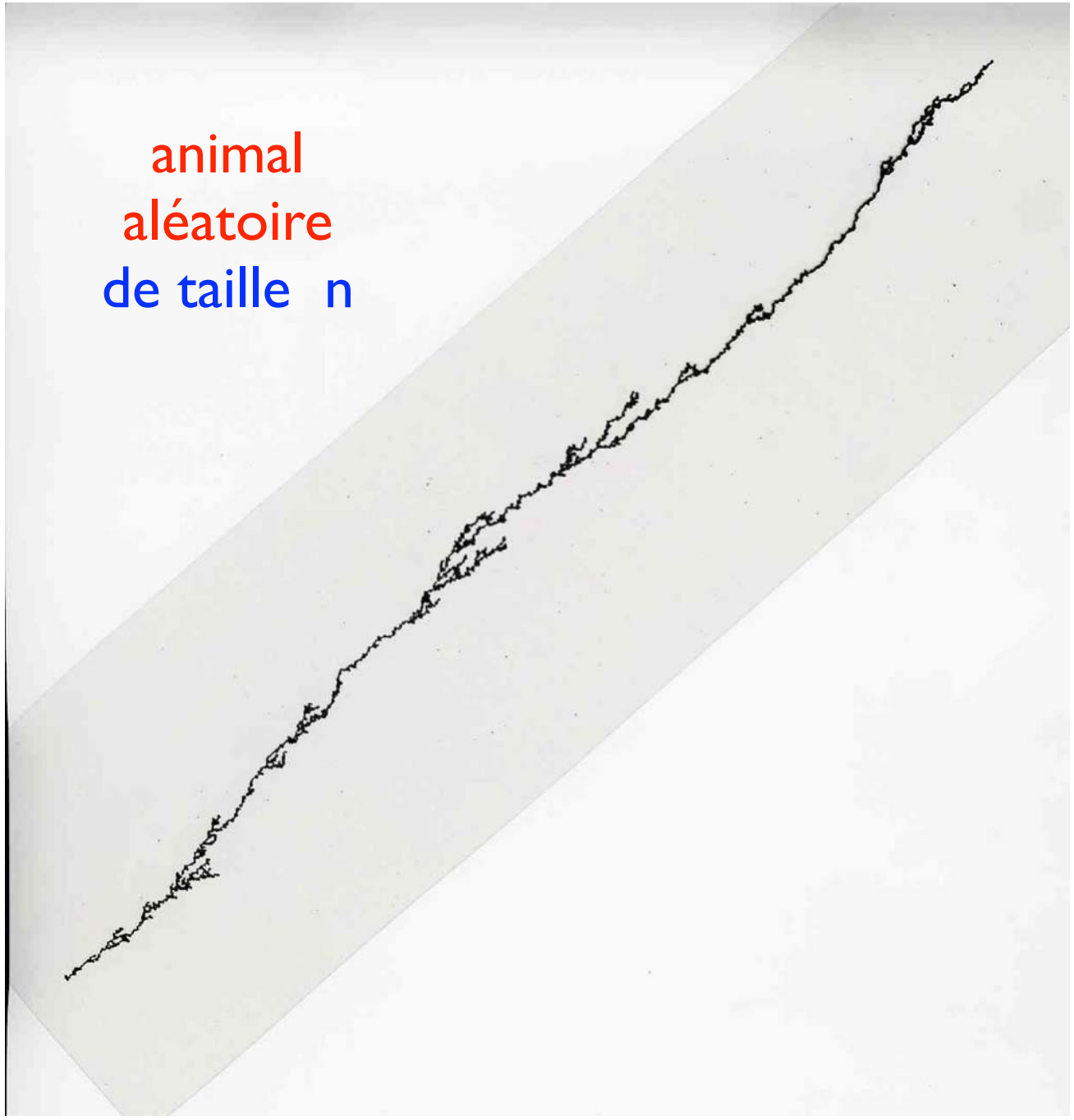
Structures discrètes aléatoires

SADA

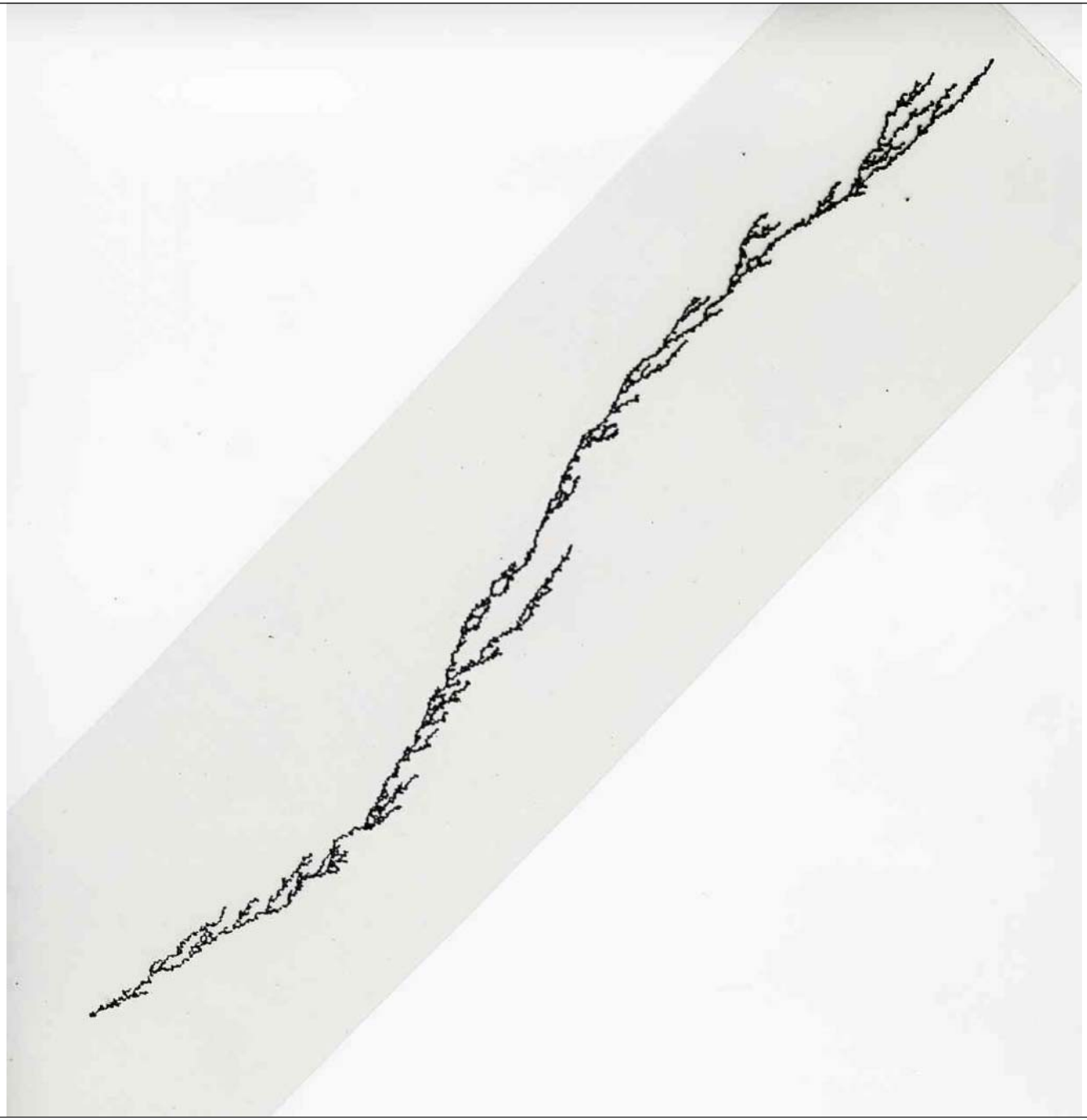
Projet blanc ANR

LaBRI, INRIA, LIX, ...

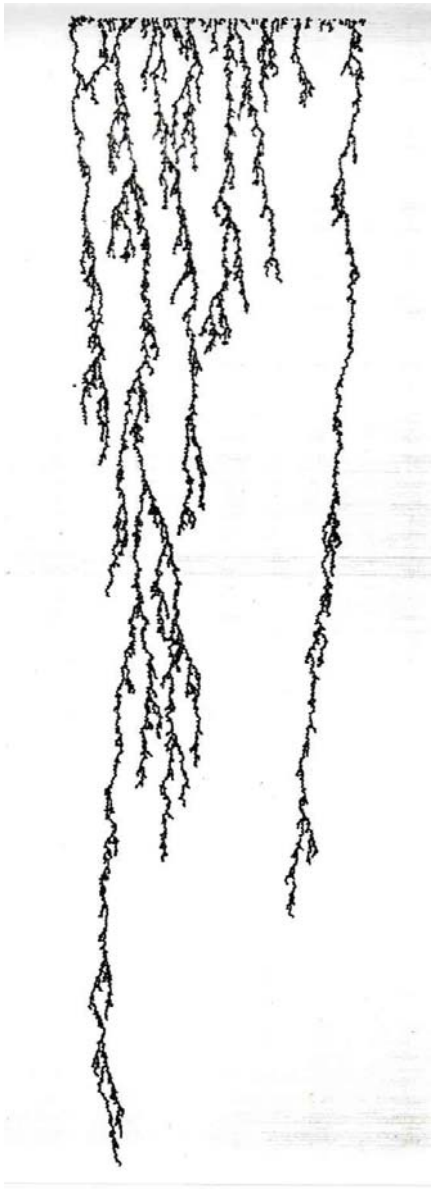
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

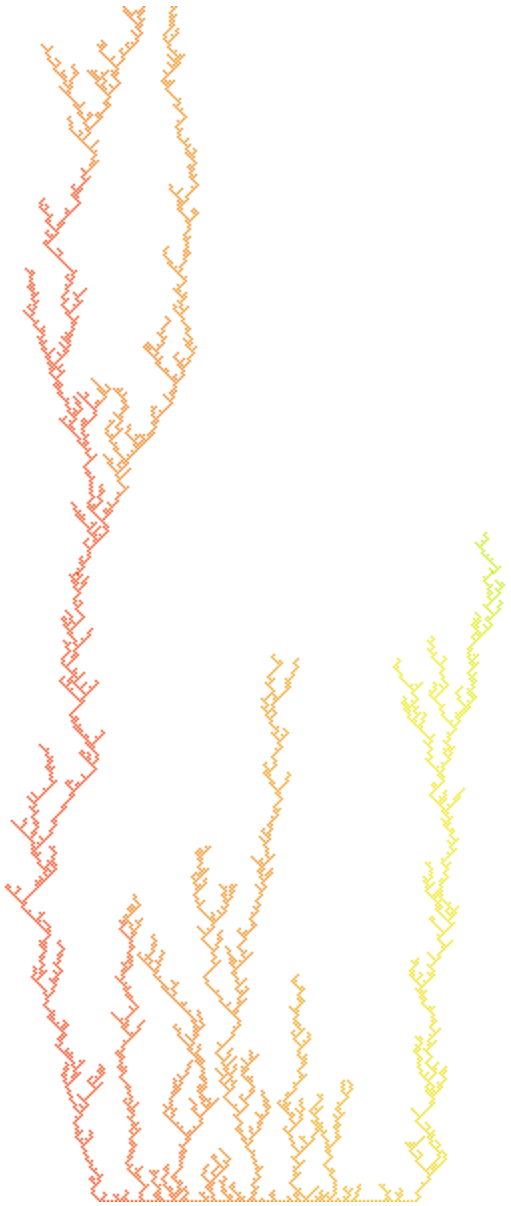


© Ministère de l'Éducation nationale



animal aléatoire
source compacte
de taille n

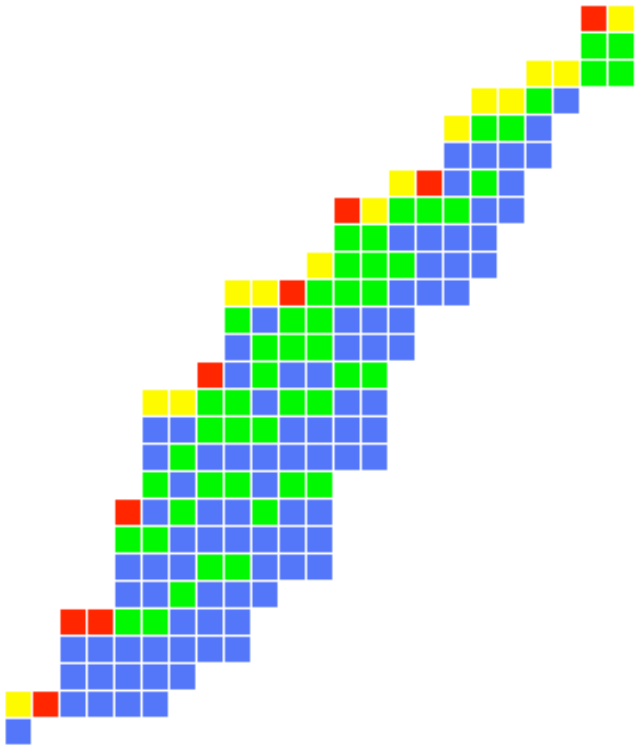
© Ministère de l'Éducation nationale



animal aléatoire source compacte de taille n

J.Betrema, LaBRI

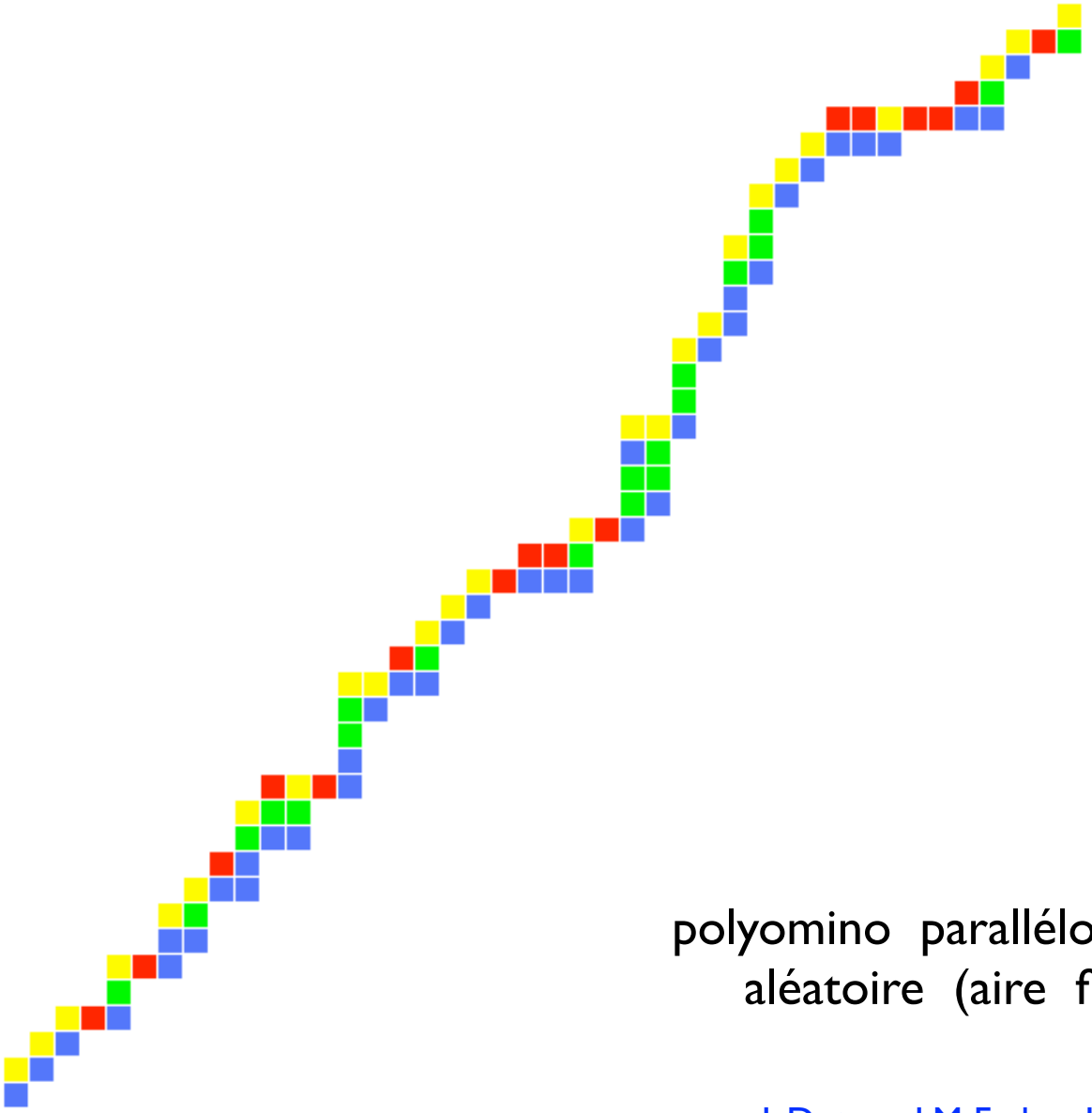
© Ministère de l'Éducation nationale



polyomino parallélogramme aléatoire (périmètre fixé)

I. Dutour, J.M Fedou, LaBRI

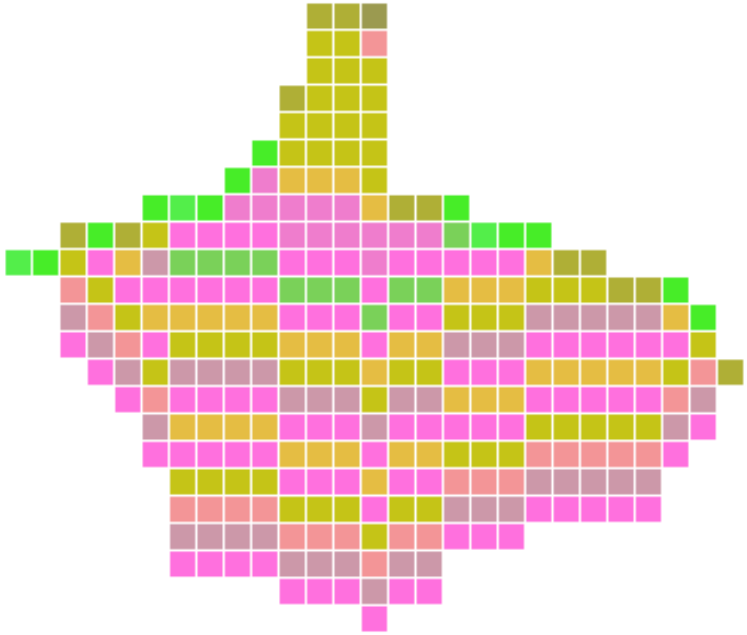
© Ministère de l'Éducation nationale



polyomino parallélogramme aléatoire (aire fixée)

I. Dutour, J.M Fedou, LaBRI

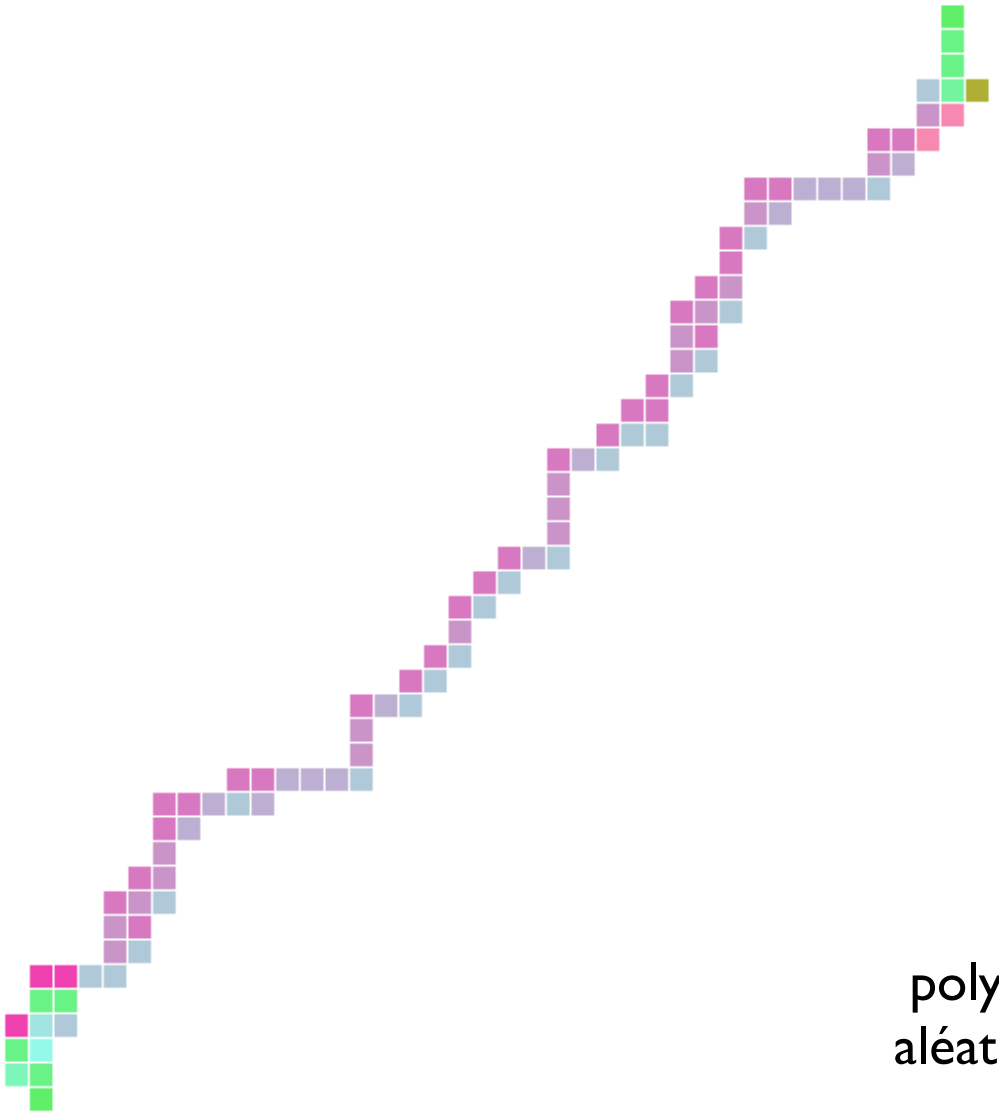
© Ministère de l'Éducation nationale



polyomino convexe
aléatoire (périmètre fixé)

I. Dutour, J.M Fedou, LaBRI

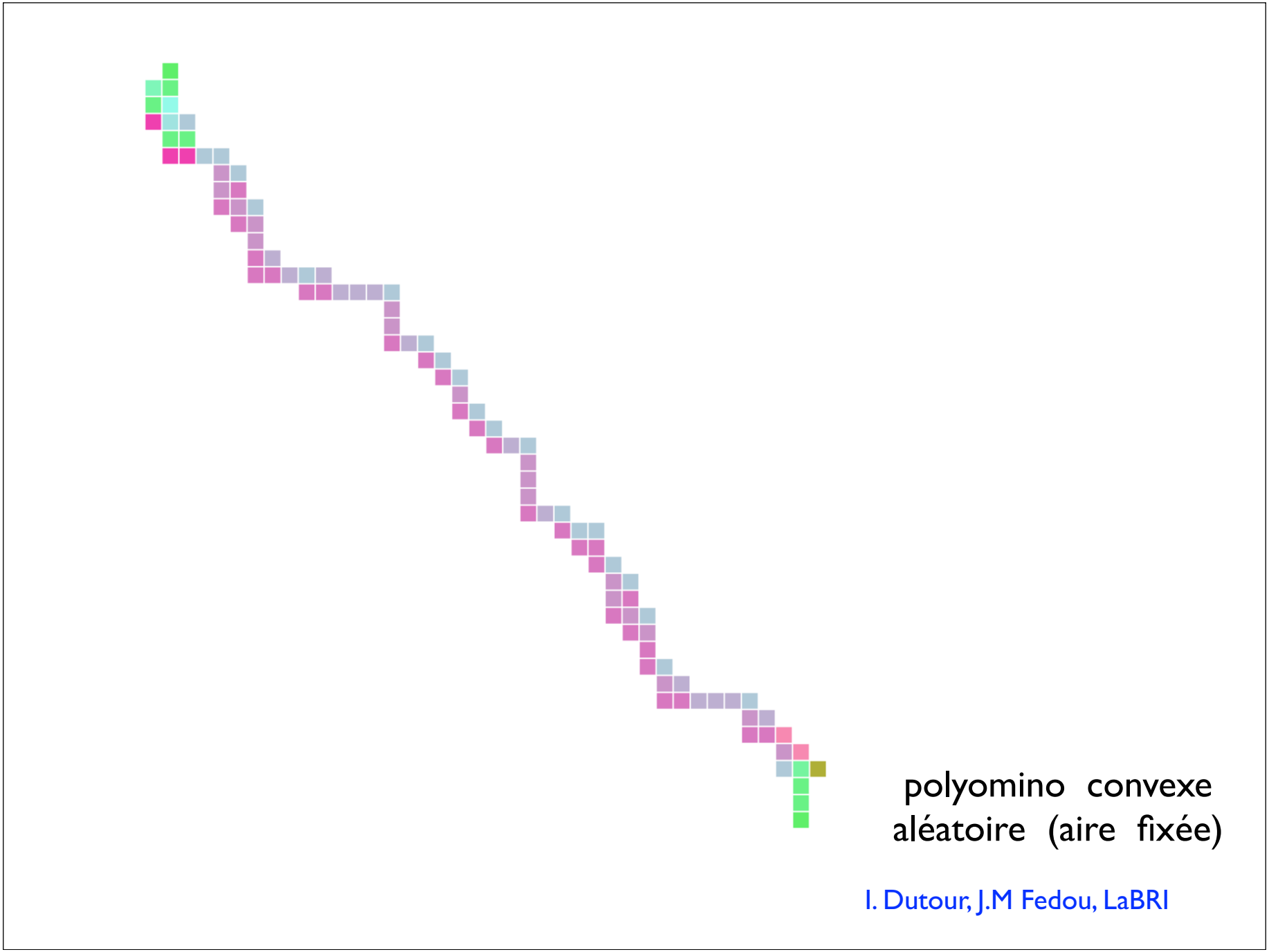
© Ministère de l'Éducation nationale



polyomino convexe
aléatoire (aire fixée)

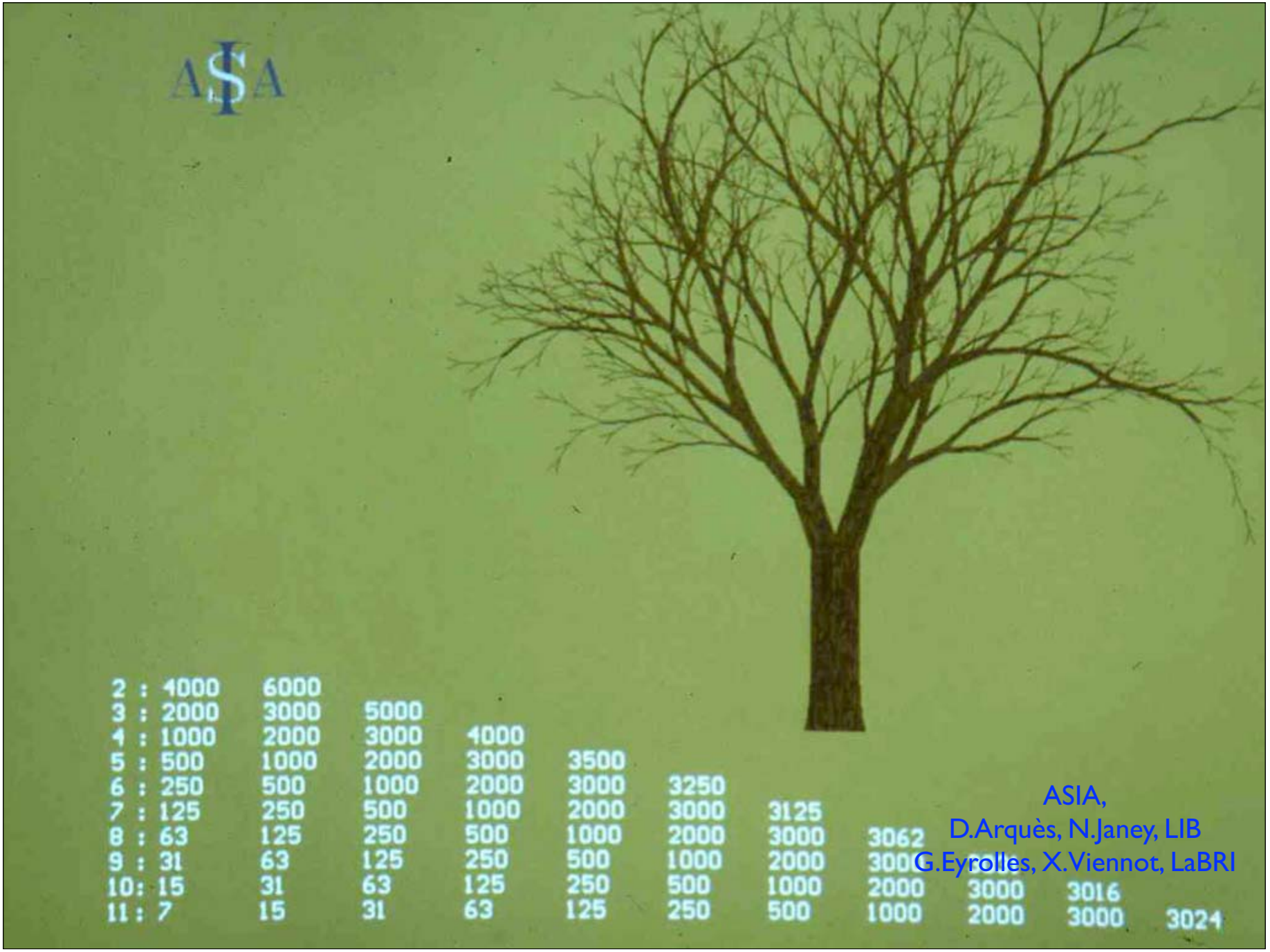
I. Dutour, J.M Fedou, LaBRI

© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

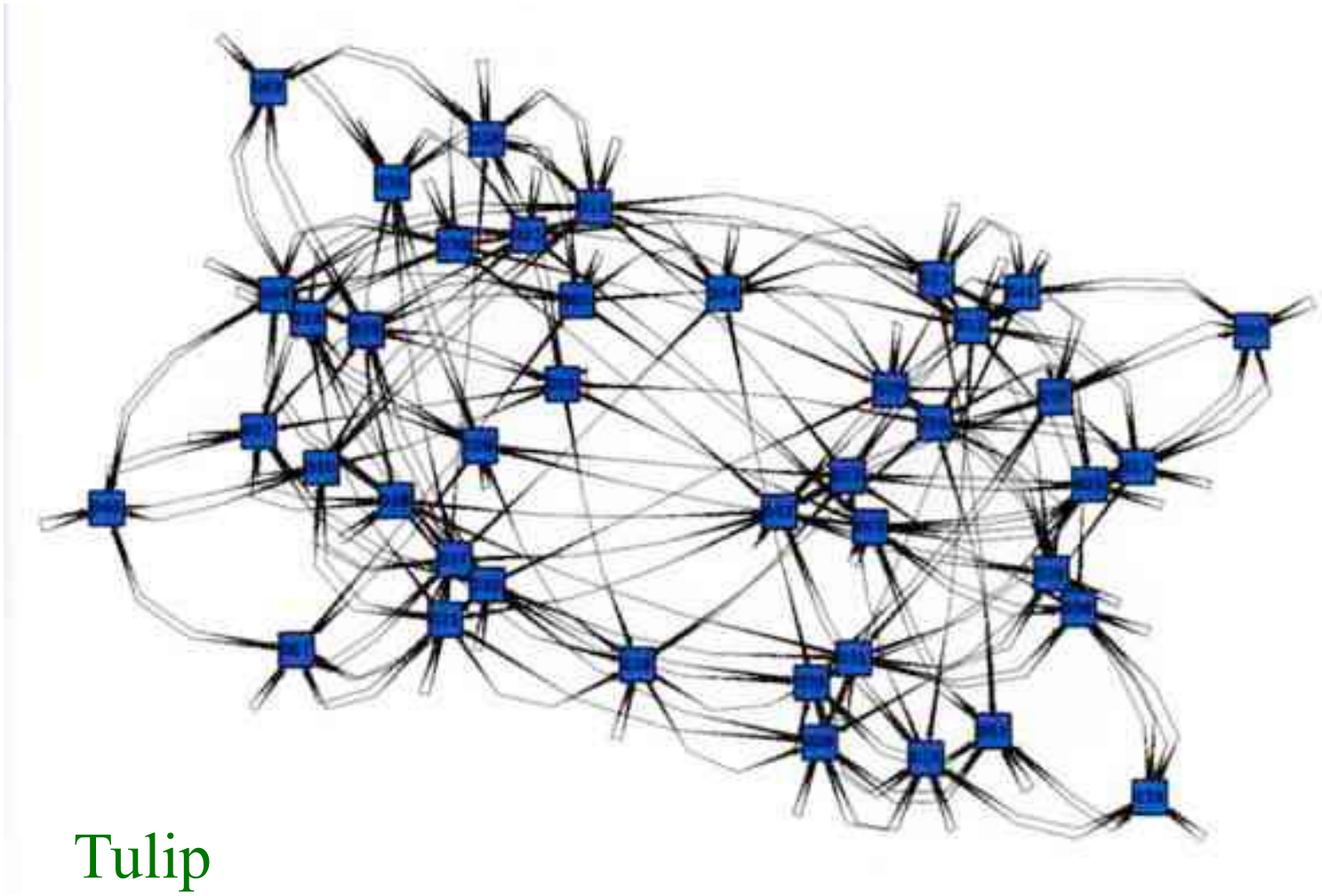
© Ministère de l'Éducation nationale



© Ministère de l'Éducation nationale

Expérimentations visuelles

© Ministère de l'Éducation nationale



Tulip

D. Auber

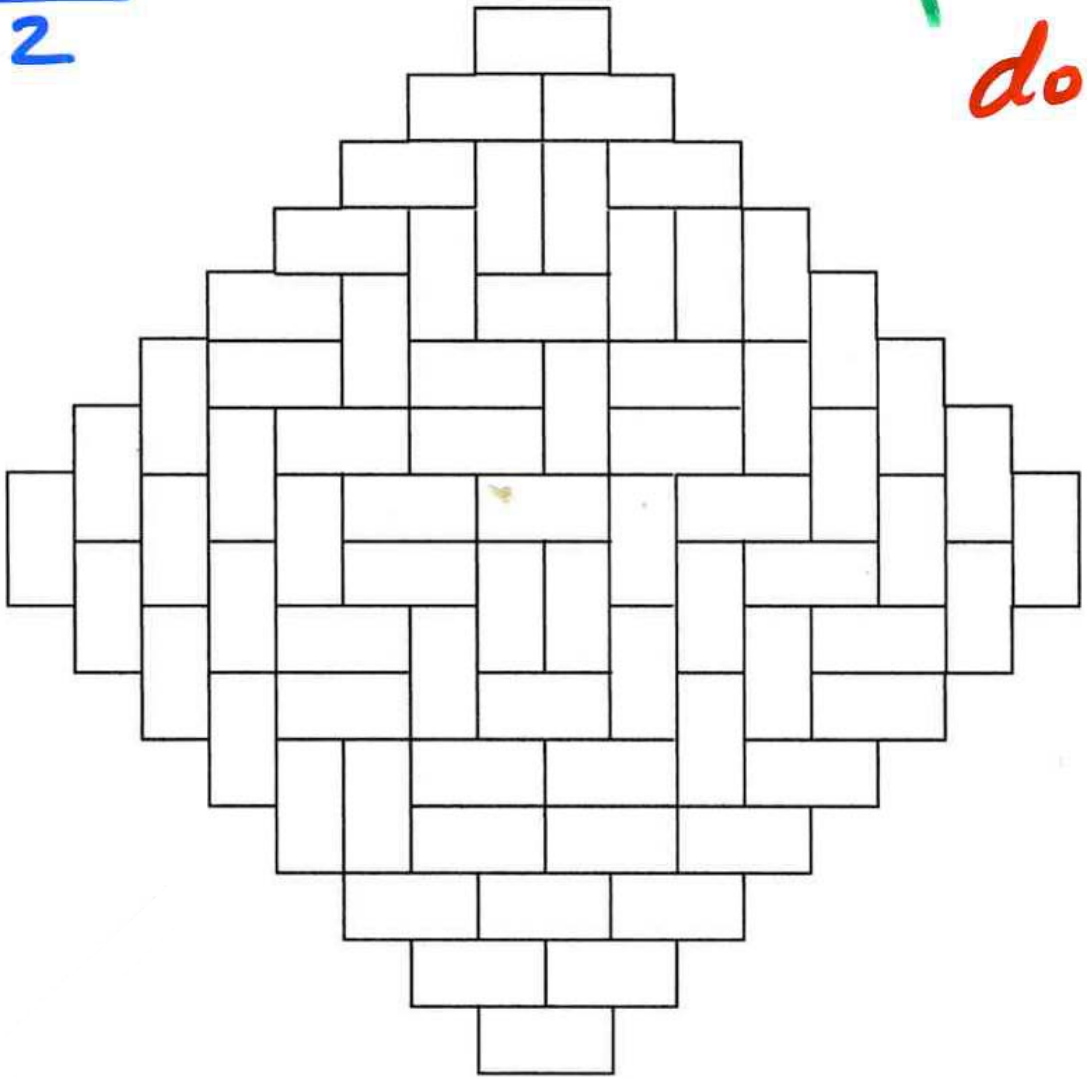
© Ministère de l'Éducation nationale

Le théorème du cercle arctique

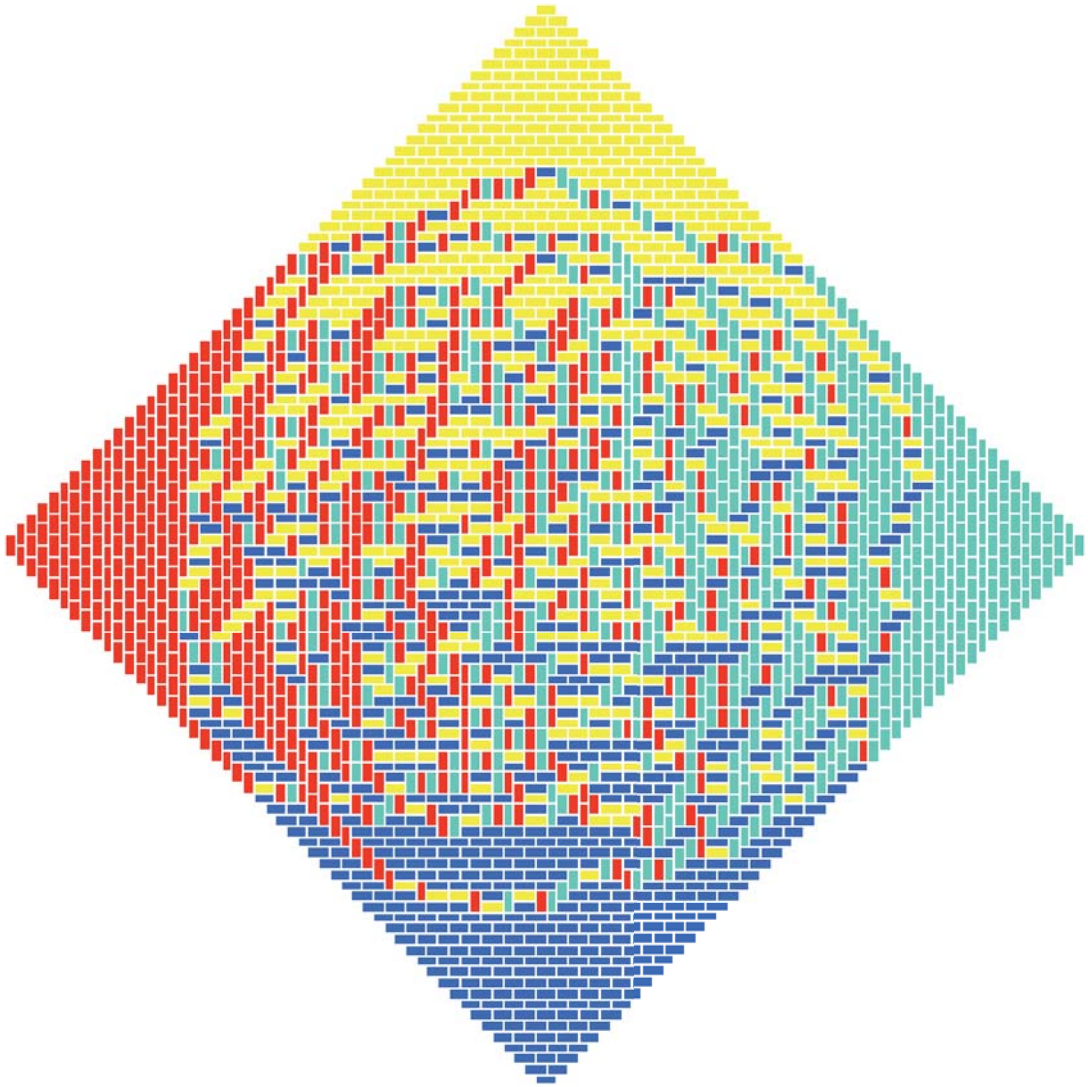
© Ministère de l'Éducation nationale

$2^{\frac{n(n-1)}{2}}$

pavage dominos

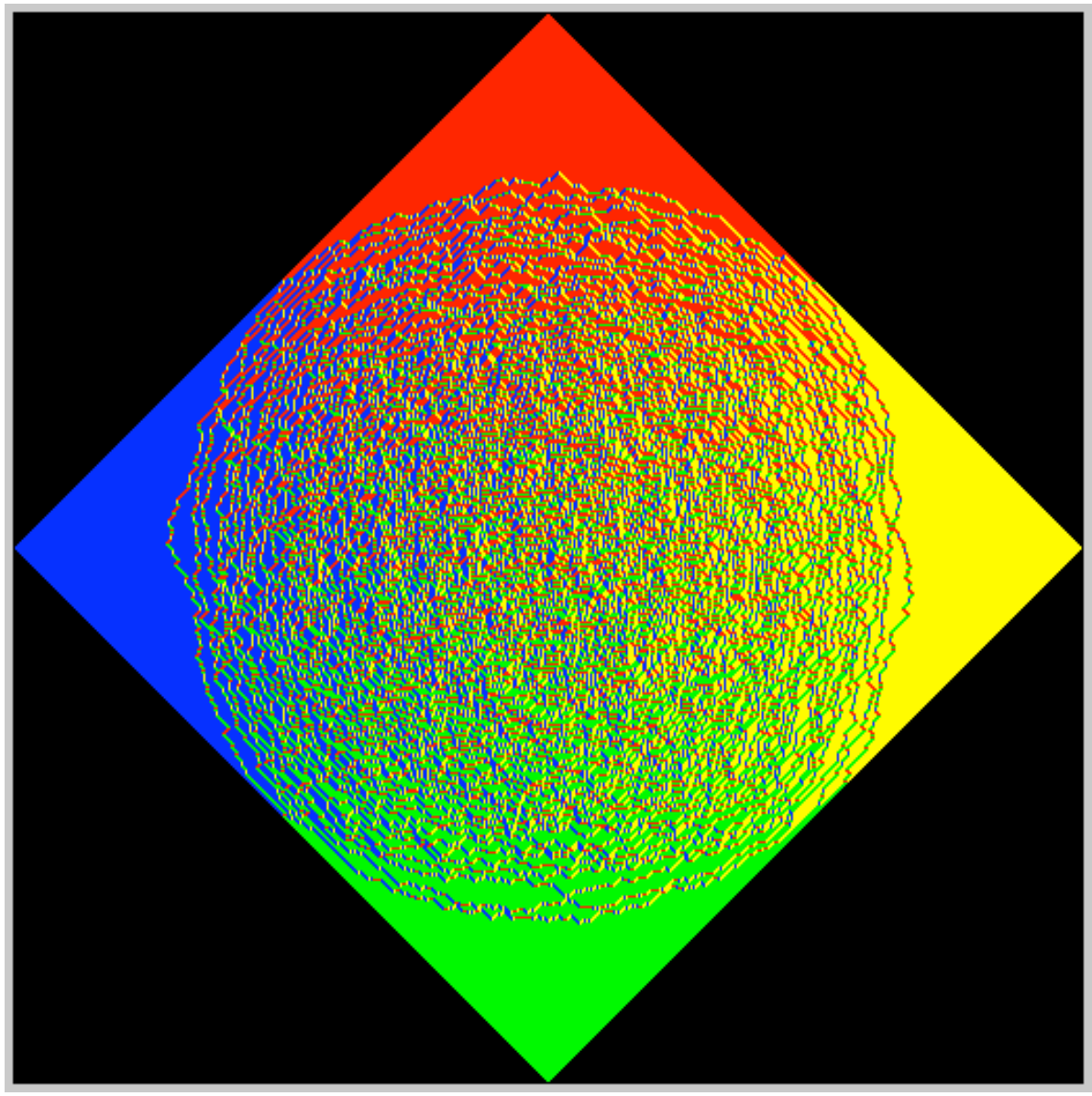


© Ministère de l'Éducation nationale



James Propp
MIT Tiling Research group
www.math.wisc.edu/~propp/tiling/www/

© Ministère de l'Éducation nationale



Chris Moore
www.santafe.edu/~moore/gallery.html

© Ministère de l'Éducation nationale

merci à tous !

www.labri.fr/perso/viennot

© Ministère de l'Éducation nationale

pour un survol pédagogique de la combinatoire contemporaine, comprenant plusieurs des sujets développés dans cet exposé, voir la leçon 6 (page 165-238) du livre:

Leçons
de mathématiques
aujourd'hui

vol 3, 2007

Cassini

X.Viennot, "Enumérons! De la combinatoire énumérative classique aux nouvelles combinatoires: bijective, algébrique, expérimentale, quantique et ... magique !"

© Ministère de l'Éducation nationale

table des matières

Combinatoire expérimentale 1

séries génératrices 11

arbres binaires 27

Une lettre oubliée d'Euler ... 35

des triangulations aux arbres binaires 46

Sloane, Plouffe 53

permutations de Baxter 62

Hipparque, 103049 ? 73

Chaînes de spins quantiques 81

MSA 89 Enumérer les MSA 94

Pavages 98

école primaire 102

théorème pentagonal 111

deviner une formule pour une série génératrice 119

polyominos convexes 123

q 138

animaux dirigés 155

SADA 170

Tulip 182

cercle arctique 184

© Ministère de l'Éducation nationale